ESSAYS ON NETWORK FORMATION

A Dissertation

by

WILLIAM GRAHAM MUELLER

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2012

Major Subject: Economics

ESSAYS ON NETWORK FORMATION

A Dissertation

by

WILLIAM GRAHAM MUELLER

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Rajiv Sarin
Committee Members,	Brit Grosskopf
	Silvana Krasteva
	Yoonsuck Choe
Head of Department,	Timothy J. Gronberg

August 2012

Major Subject: Economics

ABSTRACT

Essays on Network Formation. (August 2012)

William Graham Mueller, B.S. Oklahoma State University;

M.S., University of Illinois-Chicago

Chair of Advisory Committee: Dr. Rajiv Sarin

This dissertation contains two essays which examine the roles that individual incentives, competition, and information play in network formation. In the first essay, I examine a model in which two competing groups offer different allocation rules that may depend on the network of connections among the individuals that make up each group. I assume the existence of a single divisible good, such as a monetary prize, which will be divided amongst the members of the winning network. The probability of winning the prize will depend on the network sizes. I examine two well known allocation rules: the Myerson value and the egalitarian rule. I prove existence of equilibria and characterize the properties of the two networks. The implications of the equilibria networks for the outcome of the contest are discussed. I find that the winning probability of the network using the Myerson value has an upper bound very close to one half. There is no such upper bound for the network using the egalitarian rule.

In my second essay, I examine a dynamic model of network formation in which individuals use reinforcement learning to choose their actions. Typically, economic models of network formation assume the entire network structure to be known to all individuals involved. The introduction of reinforcement learning allows us to relax this assumption. Through the use of a state-dependent reinforcement learning rule, one may allow for varying degrees of information available to the agents. Three informational settings are examined and I determine what networks, if any, each model may converge to in the limit. The long-run behavior of each model is examined through the use of simulations and compared to one another. I find that amount and type of information agents have access plays an important role in which networks emerge when there is no dominant strategy for the agents choosing links. If there is a dominant link choosing strategy, the most efficient network structure quickly emerges in each informational setting.

Together, these essays investigate how information, incentives, and competition may affect network formation. Individual incentives in the presence of competition can create tension between an individual's social ties and the overall network size. Information plays a key role in the emergent network topologies when there are no dominate network building strategies.

ACKNOWLEDGMENTS

I want to thank my wife, Katrina, my friends, and family for the moral support they provided me during the course of my studies. I would also like to thank my advisor, Rajiv Sarin, for his support and allowing me to explore my own ideas, while gently pointing me in the right direction. My committee members Brit Grosskopf and Silvana Krasteva provided me with useful comments and moral support. Yoonsuck Choe was gracious enough to serve as my outside committee member. I would also like to thank the participants of the economic theory student seminar, and in particular, Rodrigo Velez and Alex Brown, for their comments, suggestions, and feedback they provided after my presentations. Steve Wiggins, Tim Gronberg, John Moroney, Ryo Jinnai, and Richard Anderson all gave me useful feedback during my fourth year presentation. I had many great discussions with my friends and colleagues and want to thank Nevin Aragam, Priyanka Sharma, Devika Hazra, Jeremy Nighossian, Yaojing Wang, Guanyi Ma, Yichen Gao, and Zhongwen Liang in particular.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURE Page

LIST OF TABLES

TABLE Page

CHAPTER I

INTRODUCTION

Until recently, the study of networks, for most economists, was limited to physical networks, such as railroads and other transportation networks, or network "effects" associated with technology adoption. However, more recently, economists have began to recognize the importance of networks and the structure of interactions among economic agents and or firms. Networks play an important role as information channels which individuals use learn about job opportunities, new products and technologies, and opinions on current events. Whom individuals interact with can influence outcomes involving education, criminal activity, and health related behaviors. Furthermore, the traditional view of markets as anonymous systems can fail to explain many observed empirical patterns. For these reasons, it is important to understand how and why networks form. Economics, as a discipline, can provide insight into why certain patterns of interactions form and how individuals self-organize through the use economic reasoning and useful tools from behavioral economics, decision theory, and game theory.

This dissertation contains two essays on network formation with the goal of understanding the roles information, incentives, and competition play in network formation. In the first essay, I examine the interplay between competition and individual incentives in networks. In my model, individuals are involved some network relationship and must choose between two groups offering different allocations of a divisible good, or prize, which is awarded to one group. Only one group will obtain the prize, however, each group has a certain probability of obtaining the prize. Individuals are free to choose between either group and a trade-off between individual allocation and

This dissertation follows the style of Journal of Economic Theory.

the probability of obtaining the prize is used to define an equilibrium notion. The existence and properties of the equilibrium networks are both examined.

My second essay focuses on how the information available to individuals affects network formation. This work contributes to the existing literature on network formation by modeling the individual payoffs associated with different network structures in the same way as an existing line of research, while examining a different behavioral framework which allows varying informational structures. The agents' behavior is modeled as reinforcement learning in which the agents adjust their propensities to add new connections, sever existing ones, or keep certain connections from forming based on past payoffs. Given this behavioral assumption, I examine the long-run behavior of the network for each informational structure and the effect the parameters of each model have on network topology.

CHAPTER II

COMPETING ALLOCATION RULES IN NETWORKS

A. Introduction

In many different applications, individuals might be interconnected in some networked relationship. The total value or productivity generated by these networks may depend on the network structure. Furthermore, individual compensation or allocation of the value generated may in turn depend on network structure as well. Such allocation rules were first examined in cooperation games with a fixed network structure and later, within a framework where the network structure is not permanent, but may be altered by individuals. Myerson[19] added a communication structure to a cooperative game, so that agents must communicate to be productive. He referred to such games as “communication games”. Myerson also provided an analogue of the Shapely value for communication games, which became known as the Myerson value. This seminal paper spawned a number of studies on such cooperative games including [11]. See [25] for a survey. Later, Jackson[12] extended the Myerson value to games where productivity depends explicitly on the network structure. Furthermore, they showed that the Myerson value is the unique allocation rule which satisfies certain desirable properties. The authors also characterized the axioms that uniquely determine the egalitarian rule for networks. However, this line of literature assumes that each network will receive the value produced with certainty. We can imagine a situation in which several networks are competing for a prize. Each network has some probability of winning the contest. Furthermore, each network proposes to allocate the prize among its members in some way. Individuals must now take into account both their own allocation, as well as the probability with which their network wins the prize. The DARPA Network challenge discussed in Section 1.1 is an example of such a contest.

To examine such a contest, I assume the existence of a “society” of individuals, each of whom will participate in a contest. These individuals, connected in some network structure, choose between two groups, each offering an exogenous allocation rule. Although I consider two specific allocation rules, the Myerson value and the egalitarian rule, I will mention how my results may generalize for arbitrary allocation rules. This framework is similar in spirit to [6], [5], and [8]. Hwan, et al. [8] considered a rent-seeking contest with endogenous group size and found that group sizes tend to be of equal size. The two papers of Gensemer et al., examine a framework consisting of many local environments with fixed, non-disposable, perfectly divisible endowments. Each local environment may offer different division rules and individuals are free to choose their environment. Unlike my model, these works do not allow a network structure within the groups and, furthermore, none of these works examine allocation rules in scenarios where the total allotment of rent or division of endowments depends on the outcome of a contest.

I define an equilibrium notion based on the expected value of an individual’s prize allocation, which I refer to as “expected allocation”. In an equilibrium, no individuals can increase their expected allocation by switching networks. The existence of an equilibrium will depend on the contest success function, which determines the probability of winning the prize for each network. The contest success function (CSF) is modeled typical fashion, using the functional form proposed by [26] and [23], among others, where the effort level is equal to network size. See [17] for a survey on contest related literature. I prove existence and characterize conditions under which two networks form an equilibrium for all parameter ranges used in the CSF. Surprisingly, I find that in equilibria in which both networks are non-empty, the winning probability for the group using the Myerson value is bounded above in equilibrium. In fact, the winning probability for the group using Myerson value cannot be much more than one half, and can only be greater than one half if it allocates an equal share of the prize to its members. I consider an extension of the model in which the outcome of the contest depends on the connections in each network and give an example that contrasts with the above results.

1. Motivating example

In December 2009, the Defense Advanced Research Projects Agency (DARPA) randomly placed 10 balloons in public places across the United States. The Department of Defense was interested in how social networking could be used to quickly solve large-scale problems. The person(s) who first reported the correct locations of all 10 balloons would receive a prize of \$40,000. The winning team was from MIT and identified the location of all 10 balloons in less than 9 hours. Led by physicist Riley Crane, the MIT team used a methodology they called "recursive incentive structure", which is presented in [20]. This payment structure is illustrated in Fig. 1.

The MIT team paid anyone who reported the correct location of a balloon \$2,000. If you recruited someone who knew the location a balloon you received a payment of \$1,000. If you recruited someone who in turn recruited someone who reported the correct location of a balloon, you received \$500 ... and so on. The maximum payment for finding a single balloon is $\$2000(1+\frac{1}{2}+\frac{1}{4}+\dots) = 2000(\frac{1}{1-\frac{1}{2}}) = \$4000$. Since the prize was \$40,000, the MIT team could finance the referral payments with the prize money. This payment scheme can be thought of a particular type of revenue sharing scheme, which depends on structure of the network formed by the firm. Other teams involved in the challenge offered to split the prize in different ways, including donating it all to charity, offering a chance to win Chevy Camaro as a prize, and lump sums.

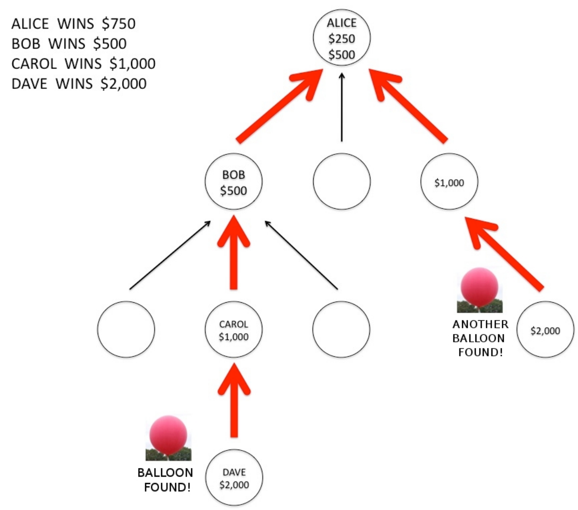

Fig. 1. Recursive incentive structure

One participant from the second place team from Georgia Tech Research Institute made some interesting comments and attributed the success of the MIT team to the recursive incentive structure they employed. This example illustrates that the allocation rules offered by different groups may affect the outcome of a contest and serves as motivation for the questions to be examined in this paper.

B. Preliminaries

1. Contest success functions

Tournaments, patent races, and rent-seeking situations have been modeled as contests in which the probability that one group or individual wins depends on the effort-level of all the players involved in the contest. The contest success function specifies how the winning probability distribution varies with the effort levels of the individuals or groups involved. Skaperdas[23] proposed several axioms a contest success function (CSF) and proved certain functional forms satisfy these axioms. The axioms

Skaperdas proposed were as follows:

i) A contest success function satisfies the conditions of a probability distribution.

ii) A player's probability of winning is increasing in their own effort, but decreasing in every other player's effort.

iii) A player's success should not depend on their identity or on the identity of the other players, but only on the efforts of the players involved.

iv) A contest success function should satisfy a basic consistency property that small contests are qualitatively similar to large contests.

v) A contest success function should include independence from irrelevant alternatives.

Two commonly used contest success functions which satisfy these five axioms are listed below. The probability that individual i (or group i) wins the contest is given by p_i where, for some $m > 0$,

$$p_i(N_i, \ldots, N_n) = \frac{N_i^m}{N_i^m + \sum_{j \neq i} N_j^m} \tag{2.1}$$

For some $k > 0$,

$$p_i(N_i, \ldots, N_n) = \frac{e^{kN_i}}{e^{kN_i} + \sum_{j \neq i} e^{kN_j}} \tag{2.2}$$

The functional form (2.1) is the only continuous functional form which satisfies axioms 1-5 and the following additional homogeneity axiom, which guarantees that the winning probability of each player is unaffected by an equally proportionate change in the effort level of each player.

$$p_i(\lambda N_1, \ldots, \lambda N_n) = p_i(N_1, \ldots, N_n) \ \forall \ \lambda > 0, i \in N$$

The functional form (2.2) is the only continuous functional form which satisfies the following property, which guarantees that the winning probability of each player depends only on the difference in the efforts among all players.

$$p_i(N_1, \ldots, N_n) = p_i(N_1 + c, \ldots, N_n + c) \ \forall c \ \in \mathbb{R}, i \in N$$

In what follows, I will assume the CSF to be of form (2.1) with the exact function to be defined in section C.

2. Networks

Before allocation rules in networks are introduced, a few preliminary definitions are needed. A network $g = (V, E)$ is represented by an undirected graph, which consists of a set of vertices, V, and the set of edges, E. Edges are represented by unordered pairs of vertices where $(i, j) \in E$ tell us that vertices $i \in V$ and $j \in V$ are connected in network g. For simplicity, let ij denote the edge (i, j). For my model it will be useful to look at the connections between a subset of vertices. This set, along with the subset of vertices, is known as a subnetwork or subgraph. The formal definition is given below.

Definition 1. *The* subnetwork on S *is denoted by* $g|_S = \{ij \in g | i, j \in S\}$ *and is found by ignoring all the connections in* g*, except those between agents in* S*.*

A subnetwork $g|_S$ in which each vertex has no additional connections to a vertex outside S is commonly referred to as a component.

Definition 2. *A* component *is a subnetwork $g|_C$ such that for each $i \in C$ if $ij \in g$, then $j \in C$. The set of all components for a network is denoted $C(g)$.*

3. Value functions and allocation rules

In network games, an agent or player is represented as a vertex of a graph. The productive level of a network is determined by its structure and is captured by a value function.

Definition 3. *A* value function *is a function $v : G(N) \to \mathbb{R}$.*

Here $G(N)$ represents that set of all networks with N nodes. The value of the empty network is usually taken to be zero (i.e., $v(\emptyset) = 0$. One prominent class of value function are those which are component additive. Component additivity precludes externalities across components of a network. Formally,

Definition 4. *A value function is* component additive *if* $\sum_{h \in C(g)} v(h) = v(g)$.

If the total value produced by the network is transferable, one must specify how it is allocated among individuals within the network. Such a rule is known as an *allocation rule* and is defined below.

Definition 5. *An* allocation rule *is a function $Y : G(N) \times \mathcal{V}(N) \to \mathbb{R}^n$ such that $\sum_i Y_i(g,v) = v(g)$ for all v and g, where $\mathcal{V}(N)$ denotes the set of all possible value functions. $Y_i(g,v)$ represents the allocation agent i receives under allocation rule Y.*

4. Examples of allocation rules

One popular example of an allocation rule is the *egalitarian allocation rule* denoted Y^e and defined as follows: $Y_i^e(g,v) = \dfrac{v(g)}{n}$. This rule shares the value produced by the network equally among all members of the network. Another interesting example of an allocation rule is the Myerson Value. The Myerson value was first proposed in [19] and later extended to a network games setting in [9]. It is based on the Shapely value of an associated cooperative game and is defined below.

$$Y_i^{MV}(g,v) = \sum_{S \subset N \setminus \{i\}} (v(g|S \cup i) - v(g|S)) \frac{\#S!(n - \#S - 1)!}{n!} \tag{2.3}$$

Here S is a subset of the agents involved in the network, $g|S = \{\{i|ij \in g\}|j \in S\}$ is the subnetwork restricted to S, and $v(g|S)$ is the value assigned to this network by the value function. Like the Shapely value, the Myerson value may be interpreted in terms of random arrival time. Imagine the network being formed over time, with each individual demanding their marginal contribution, $v(g|S \cup i) - v(g|S)$. Averaging across all possible permutations gives us the Myerson value.

5. Properties of allocation rules

The following two properties uniquely characterize the Myerson value in network games. The first property, known as component balance requires the value produced by a component to be allocated among the individuals that comprise the component, for any component additive value function. If the value function is not component additive, this condition will be violated.

Definition 6. *An allocation rule Y satisfies* component balance *if* $\sum_{i \in S} Y_i(g,v) = v(g|S)$

The second condition known as equal bargaining power, requires that two individuals in a network will benefit or suffer equally with the addition or deletion of a connection between the two agents.

Definition 7. *An allocation rule Y satisfies* equal bargaining power *if for all v, g, and, $ij \in g$*

$$Y_i(g,v) - Y_i(g-ij,v) = Y_j(g,v) - Y_j(g-ij,v)$$

Jackson[9] proved the following extension of Myerson's work characterizing the rules that satisfy the above criteria:

Proposition 1. *[9] Y satisfies component balance and equal bargaining power if and only if $Y(g,v) = Y^{MV}(g,v)$ for all $g \in G$ and any component additive v.*

This proposition implies that any allocation rule which satisfies both component balance and equal bargaining power requires that two players share the value of a single link. Furthermore, provided that the value of a network is the sum of the value of the individual links, that is $v(g) = \sum_{ij \in g} v\{ij\}$. Then the Myerson value is equivalent to:

$$Y_i(g,v) = \frac{\sum_{ij \in g} v(\{ij\})}{2}$$

Throughout this paper, I will restrict my attention to value functions which assign a value to each network equal to the sum of the value of the individual links throughout this paper. Under this restricted domain, the Myerson value is equivalent to the positional value, another popular allocation rule in network games.

C. Model

There are N agents located at the vertices of a graph $g = (V,E)$, which represents their social network. I assume the existence of single divisible good, "the prize" which is valued equally by each agent. There is a value function $v : G(N) \to \mathbb{R}$. I will assume v is component additive. In other words, the value agents in isolated components contribute should not be affected by the value produced by a different isolated component. This assumption seems reasonable for the model. Furthermore, I will assume that $v(g) = \sum_{ij \in g} v\{ij\}$. This assumption allows the Myerson value to be easily calculated. It is important note that in my model, the value function does not represent a value to be allocated among individuals, but can be thought of how well a network communicates or how much individuals "work'. The value function provides a basis for the allocation of the prize.

The set of agents is partitioned into two sets A and B which identify the individuals associated with each competing network. From this partition, the network is partitioned into two subnetworks, $g|_A$ and $g|_B$, which capture the network structure of the two competing groups. Each group has an allocation rule, denoted by Y^A and Y^B. These allocation rules are defined over the subnetworks $g|_A$ and $g|_B$. The allocation to i is given by: $\frac{Y_i^A(g|_A, v)}{v(g|_A)}$ if $i \in A$ and $\frac{Y_i^B(g|_B, v)}{v(g|_B)}$ if $i \in B$ where Y^A and Y^B may be different. The actual allocation is given by a normalized version of the allocation rule, in order to ensure the groups only divide the amount of the prize. For simplification, I will take Y^A and Y^B to represent these normalized allocation rules. Throughout this paper, I will consider two well-known allocation rules: the Myerson value and the egalitarian rule, though I will point out results that generalize for any allocation rule. The probability of winning the contest depends on the number of agents in each subnetwork; $|A|$ and $|B|$ respectively. The probability that group A wins the contest is given by the following:

$$p_A(A, B) = \frac{|A|^m}{|A|^m + |B|^m} \tag{2.4}$$

where $m > 0$ is the parameter that determines the curvature of the CSF. Since

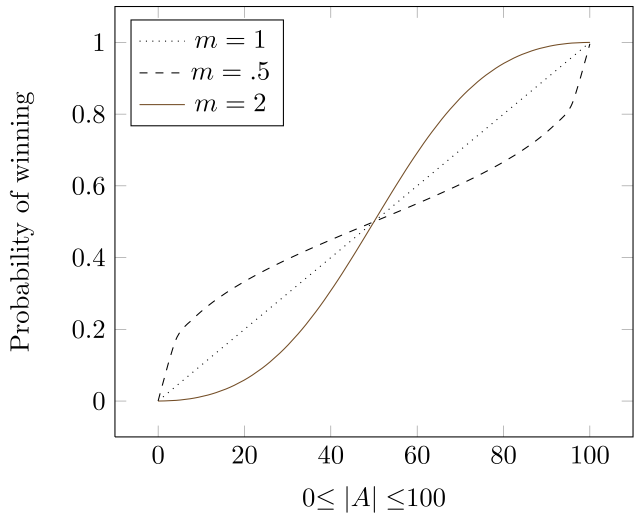

Fig. 2. Effect of m on the CSF

$N = |A| + |B|$, the CSF is just a function of one variable and the parameters m and N. The role m plays in the winning probability for a group is illustrated in Fig. 2.

Due to the fact that agents will not receive a share of the prize with certainty, they must take into account both the size of their individual allocation and the probability with which their network wins the prize. I define a notion of stability of the partition of a network g into two subnetworks $g|_A$ and $g|_B$ where network A uses allocation rule Y^A and network B uses allocation rule Y^B in the following manner. Two competing networks form an equilibrium partition if no agent wants to unilaterally switch their allegiance, that is, they cannot increase their expected allocation by defecting to the other group. Formally,

Definition 8. *Given contest success function p, two subnetworks $g|_A$ and $g|_B$ form an equilibrium partition respect to allocation rules Y^A and Y^B if the following inequalities hold :*

i) If $i \in A$ *then* $p_A(A,B)Y_i^A(g|_A, v) \geq p_B(A \backslash \{i\}, B \bigcup \{i\})Y_i^B(g|_{B \bigcup i}, v)$

ii) If $j \in B$ *then* $p_B(A,B)Y_j^B(g|_B, v) \geq p_A(A \bigcup \{i\}, B \backslash \{i\})Y_j^A(g|_{A \bigcup j}, v)$

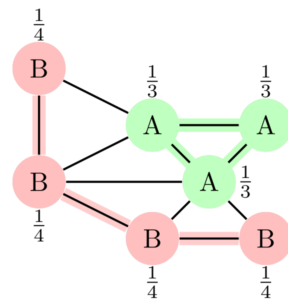

Fig. 3. An equilibrium partition with $m = 1$ in CSF 1

Fig. 3 shows an equilibrium partition of a network with $N = 7$ using CSF (2.1) with the parameter value $m = 1$. In this example, the expected allocation for each individual is $\frac{1}{7}$, though the proposed allocations are different for each group. This example illustrates the trade-off between individual allocation and network size. Individuals in smaller networks obtain a larger allocation of the prize, but at the cost of a lower winning probability.

From now on, I will assume $Y^A = Y^{MV}$ and $Y^B = Y^e$. The equilibrium defining inequalities become the following:

$$\text{For each } i \in A, \frac{\displaystyle\sum_{ik \in g|A, k \neq i} v\{ik\}}{2v(g|A)} \frac{|A|^m}{|A|^m + |B|^m} \geq \frac{1}{|B|+1} \frac{(|B|+1)^m}{(|A|-1)^m + (|B|+1)^m} \tag{2.5}$$

$$\text{For each } j \in B, \; \frac{1}{|B|}\frac{|B|^m}{|A|^m + |B|^m} \geq \frac{\displaystyle\sum_{jk \in g|A\cup\{j\}, k\neq j} v\{jk\}}{2v(g|A\cup\{j\})} \frac{(|A|+1)^m}{(|A|+1)^m + (|B|-1)^m} \tag{2.6}$$

If the value function assigns the same value to each link, the Myerson value has a nice interpretation in terms of the vertex degree, which is the number of edges of agent i participates in the network g. Formally, $v\{ij\} = v$ for any $ij \in g$, then each agent i receives an allocation of $Y_i^{MV}(g, v) = \frac{k_i}{\sum_{j\in N} k_j}$, under the normalized Myerson value, where k_i denotes the vertex degree This follows from the fact that the value function is component additive, Proposition 1, and the assumption on the value function. To see this, first note that $Y_i^{MV}(g, v) = \sum_{ij\in g} \frac{v}{2}$. Therefore, the allocation for agent i is just half the total number of links agent i participates in (which is defined as the *degree*, denoted k_i, of agent i in g) times v. In order to normalize the total allocation to sum to one, this value must be divided by the total value of the network $v(g)$. Therefore, $Y_i^{MV}(g, v) = \frac{k_i}{2v(g)}$. Since the value function is component additive, the total value of the network is simply the total number of links. This is equivalent to $\frac{\sum_{j\in g} k_j}{2}$, since each link connects two agents, the summation of the agents' degrees is twice the total number of links in the network. Finally, one obtains $Y_i^{MV}(g, v) = \frac{k_i}{\sum_{j\in g} k_j}$. This assumption allows the Myerson value to be interpreted in terms of the geometric properties of the network. Furthermore, imposing this assumption will allow us to interpret the geometric properties of the equilibrium networks. The following definitions will be useful for characterizing our results:

Definition 9. *The* density *of a subgraph on vertex set A is defined as the ratio of edges to vertices in a subgraph, $\frac{|E(A)|}{|A|}$ where $E(A)$ is the set of edges in the subgraph induced by A. This may be written as $\frac{\sum_{i\in A} k_i}{2|A|}$ since the sum of the vertex degrees is* *twice the number of edges.*

Definition 10. *A* clique *is a subgraph in which each vertex is connected to all others. That is, for a clique of size $|A|$ each vertex has a degree of $|A| - 1$.*

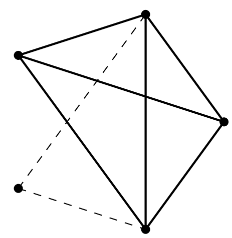

Fig. 4. Graph with a clique of size 4

These definitions are illustrated in Fig. 4. The subgraph highled is a clique of size 4 with density $\frac{3}{2}$. In general, one may express the density of a clique as $\frac{N-1}{2}$ where N is the size of the clique.

1. A special case: The complete network

When the underlying network is the complete network, the two allocation rules are equivalent. This can be seen from the fact that in any complete graph of size N, each vertex has a degree of $N - 1$. Using this fact, $\frac{k_i}{\sum_{j \in g} k_j} = \frac{1}{N}$. Furthermore, in any subnetwork of the complete network, each vertex has an equal number of connections. Each individual's allocation under the Myerson value applied to such a subnetwork will be equal to $\frac{1}{|A|}$. Taking this into account, it follows that whenever the underlying network is the complete network, both groups are offering an equal split of the prize for their members. Therefore, this must considered this as a special case.

The following results show the existence of equilibrium partitions for all values of the parameter m. For the case in which $m > 1$, the only equilibria involve the grand alliance (i.e. one of the two subnetworks is empty). Whenever $m = 1$, any partition is an equilibrium. The most interesting case occurs whenever $m < 1$ and the results show that if the two allocation rules are equivalent, the two equilibrium group sizes will be very close or equal. This result is similar to those obtained by [8]. Some simple intuition may be gained by thinking about two limiting cases for the contest success function: $m = 0$ and $m = \infty$. If $m = 0$, the contest success function assigns a winning probability $\frac{1}{2}$ to each network The winning probability is independent of network size and each individual would only care the size of their proposed allocation. Compared to the larger group, individuals in smaller group will always give a larger share of the prize. This creates an incentive for all individuals to move to the smaller network. Therefore, in any equilibrium the subnetwork sizes should be the same or very close. If $m = \infty$, then the larger network will win with probability one. Individuals always prefer to move to the larger network since individuals in the smaller subnetwork have an expected payoff of zero. Therefore, the only equilibria involves the grand alliance.

Proposition 2. *Suppose g is the complete graph and $m > 1$ in the CSF. The only equilibrium partitions are those in which $|A| = N$ or $|B| = N$.*

Proof. If $j \in B$ then $\frac{v}{|B|}p_B(A,B) \geq \frac{k_j^{A\bigcup\{j\}}v}{\sum_{i\in A\bigcup\{j\}} k_i^{A\bigcup\{i\}}}p_A(A+1,B-1)$ where $k_i^{A\bigcup\{i\}}$ is the degree of node i in $g|_{A\cup\{j\}}$. Since Γ is the complete network $k_i^A = |A|$. Furthermore, $\sum_{i\in A\bigcup\{j\}} k_i^{A\cup\{j\}} = (|A|+1)(|A|)$. Using this fact,

$$\frac{|A|+1}{|B|} \geq \frac{p_A(A+1,B-1)}{p_B(A,B)} \tag{2.7}$$

Now, if $j \in A$, then $\frac{k_i^A v}{\sum_{j\in A} k_j^A}p_A(A,B) \geq \frac{v}{|B|+1}p_B(A-1,B+1)$. Again, using the fact that Γ is the complete network. I have $\frac{k_i^A}{\sum_{j\in A} k_j^A} = \frac{|A|-1}{(|A|)(|A|-1)}$. Therefore,

$$\frac{|B|+1}{|A|} \geq \frac{p_B(A-1, B+1)}{p_A(A, B)} \tag{2.8}$$

If one substitutes the contest success function given by Eq. (2.1), into Eq. (2.5) and Eq. (2.6) the following is obtained:

$$|A| + 1 + (|A|+1)^{1-m}|B-1|^m \geq |B| + |B|^{1-m}(|A|)^m \tag{2.9}$$

$$|B| + 1 + (|B|+1)^{1-m}|A-1|^m \geq |A| + |A|^{1-m}(|B|)^m \tag{2.10}$$

If $m = 1$, then both inequalities reduce to $|A| + |B| \geq |A| + |B|$, which is trivially satisified for any partition. □

Proposition 3. *Suppose g is the complete graph and and $m = 1$ in the CSF. Then any partition of g into two subnetworks $g|A$ and $g|B$ forms an equilibrium.*

Proof. Now assume $m > 1$. If $|A| + 1 = |B|$, then inequality (5) is satisfied. Since $|A| + 1 = |B|$, it is true that $|A| < |B| + 1$. Therefore, the following must be true in order for (6) to be violated:

$$(|B|+1)^{1-m}|A-1|^m < |A|^{1-m}(|B|)^m \tag{2.11}$$

which one may rewrite as:

$$\frac{|B|+1}{|A|} < \frac{(|B|+1)^m|B|^m}{|A|^m(|A|-1)^m} \tag{2.12}$$

Since $|A|+1=|B|$ and $m>1$, it follows that $\frac{(|B|+1)^m}{|A|^m} > \frac{|B|+1}{|A|}$ and $\frac{(|B|)^m}{(|A|-1)^m} > 1$. Therefore, inequality (6) is never satisfied. If $|A|+1<|B|$, then clearly $|B|+1>|A|$ and the same argument shows inequality (6) is not satisfied. If $|A|+1>|B|$, then $|B|+1 \geq |A|$ and again, the argument is symmetric. This shows that if $m>1$ both inequalities cannot be satisfied simultaneously and there are no stable components with both $|A|>0$ and $|B|>0$. □

If $m=1$, the probability is linear in group size and equal to $\frac{|A|}{N}$ for a group of size $|A|$. Therefore, the expected allocation is constant and equal to $\frac{1}{N}$ for any subgraph. Clearly, each agent in the resulting subgraph of an arbitrary partition have the same expected allocation.

Proposition 4. *Suppose g is the complete graph and $m<1$ in the CSF. Then there exist equilibrium partitions such that $|B|-1 \leq |A| \leq |B|+1$.*

Proof. Suppose $m<1$. If $|A|+1>|B|$ and $m<1$, both $\frac{(|A|+1)^m}{|B|^m}$ and $\frac{(|A|)^m}{(|B|-1)^m}$ are smaller in magnitude than $\frac{|A|+1}{|B|}$. Furthermore, if $|A|+1>|B|$, then $|B| \leq |A|$. There are three possibilities for the relationship between $|B|+1$ and $|A|$.

Case 1: $|B|+1=|A|$. In this case inequality (2.5) is satisfied.

Case 2: $|B|+1>|A|$. This implies $|A|=|B|$. Then inequality (2.6) may be satisfied since $m<1$ implies both both $\frac{(|B|+1)^m}{|A|^m}$ and $\frac{(|B|)^m}{(|A|-1)^m}$ are smaller in magnitude, which implies the product is smaller as well.

Case 3: $|B|+1<|A|$. Inequality (2.6) will never be satisfied since $\frac{|B|+1}{|A|}<1$ which implies both $\frac{(|B|+1)^m}{|A|^m}$ and $\frac{(|B|)^m}{(|A|-1)^m}$ are larger in magnitude, which implies the product is larger as well.

Therefore, there always exist equilibria in which $|B|-1 \leq |A| \leq |B|+1$. □

2. The general case

The complete network is completely symmetric and introducing asymmetry to the network will cause the two allocation rules to be different and in turn affect the equilibria. In what follows, the underlying network g is assumed to be different than the complete network. The following propositions prove the existence of equilibrium partitions for all ranges of the parameter m in the CSF and characterize some of the properties of the equilibrium subnetworks. The results show that the presence of a competing allocation rule that offers to split the prize equally keeps the other group "honest". I mean this in the sense that each individual's allocation cannot be much different than what it would be under the egalitarian rule. In fact, it is shown that the group offering the Myerson value may only have a winning probability greater than one half if it allocates an equal share of the prize to each individual. Again, some intuition for these results may gained by thinking about the limiting cases: $m = 0$ and $m = \infty$. If $m = 0$, the contest success function assigns $\frac{1}{2}$ to each group. The winning probability is independent of network size and each individual would only consider the allocation they would receive. An individual would generally favor the smaller group, unless the smaller group is using the Myerson value and the agent has too few connections. This is the source of the asymmetry in sizes of the two networks. If the group sizes are equal or very close and if each individual in group A has the same number of connections, then there will be no incentive to switch networks. If $m = \infty$, then the larger network wins the contest with probability one. The expected allocation is zero for any agent in the smaller network and the incentive is always join the larger network. Therefore, the only possible equilibrium involving everyone joining the same network. These two examples illustrate the incentives involved for small and large m.

Proposition 5. *Let g be an arbitrary network and suppose $m > 1$ in the CSF. The grand alliance $|A| = N$ is always an equilibrium partition whenever*

$$m \geq 1 + \frac{\log\left(\sum_{j\in A} k_j - k_{min}\right) - \log\left(k_{min}\right)}{\log\left(N-1\right)}$$

where k_{min} is the minimum vertex degree in g, while the grand alliance $|B| = N$ is always stable.

Proof. Since $m = 1$, the following inequality holds for each $i \in A$

$$\frac{Y_A^i(g|A)}{v(g|A)} \frac{|A|}{|A|+|B|} \geq \frac{1}{|B|+1} \frac{|B|+1}{|A|+|B|} \tag{2.13}$$

This yields

$$\frac{Y_A^i(g|A)}{v(g|A)} \geq \frac{1}{|A|} \tag{2.14}$$

Note that $\sum_{i\in A} \frac{Y_A^i(g|A)}{v(g|A)} = 1$, which means if $\frac{Y_A^i(g|A)}{v(g|A)} > \frac{1}{|A|}$ for some $i \in A$, then $\frac{Y_A^j(g|A)}{v(g|A)} < \frac{1}{|A|}$ for some $j \in A$. However, this violates the stability inequality for group A. Therefore, $\frac{Y_A^i(g|A)}{v(g|A)} = \frac{1}{|A|}$ for any stable subgraph $g|A$. $\square$

The asymmetry of the result is due to the fact that the agent with the smallest allocation in group A may have an incentive to join group B despite the very small probability they would win the contest as a single individual. Furthermore, an agent with no connections will receive an allocation of zero under the Myerson value and will always prefer the egalitarian rule. The lower bound for m is a sufficient condition for the grand alliance $|A| = N$ to be an equilibria. This bound depends on the minimum number of connections an agent in g has. A slightly more intuitive result is summarized in the following corollary.

Corollary 1. *The grand alliance $|A| = N$ is always an equilibrium partition whenever $m \geq 2$ and g has no isolated vertices.*

This result is simply a special case of the above proposition. □

Proposition 6. *Let g be an arbitrary network and suppose $m = 1$ in the CSF. The only equilibrium partition with both $|A| \neq \emptyset$ and $|B| \neq \emptyset$ must have $Y^{MV}(g|A, v) = \frac{1}{|A|}$. Consequently, each agent in group A must have an equal number of connections.*

Proof. For the case when $|A| = N$, the following must be satisfied for each $i \in A$:

$$\frac{k_i^A}{\sum_{j \in A} k_j^A} \frac{N^m}{N^m} \geq \frac{1}{1} \frac{(1)^m}{(N-1)^m + (1)^m} \tag{2.15}$$

which can be rewritten as:

$$\frac{k_i^A}{\sum_{j \in A} k_j^A} \geq \frac{1}{(N-1)^m + 1} \tag{2.16}$$

Clearly, this will not hold for all $i \in A$ unless $k_i^A > 0$. Thus, there can be no isolated vertices (degree zero) in g. Furthermore, this inequality must hold for all agents $i \in A$. Minimizing the left hand side yields:

$$\frac{1}{1 + (N-1)(N-2)} \geq \frac{1}{(N-1)^m + 1} \tag{2.17}$$

Simplifying, and taking the logarithm of both sides one obtains: $m \geq 1 + \frac{\log N-2}{\log N-1}$. The grand alliance $|A| = N$ may be an equilibrium for smaller values of m, however, it is only guaranteed as an equilibrium if it is larger than than the stated value.

For the case when $|B| = N$, the following must be satisfied for each $j \in B$:

$$\frac{1}{B}\frac{N^m}{N^m} \geq \frac{1}{1}\frac{(1)^m}{(N-1)^m + (1)^m} \tag{2.18}$$

this simplifies to the following:

$$\frac{1}{N} \geq \frac{1}{(N-1)^m + 1} \tag{2.19}$$

This will hold whenever $(N-1)^m \geq N-1$ which is true whenever $m \geq 1$. $\square$

In any equilibrium partition, the allocation rule for group A must allocate an equal share of the prize to each individual. Based on our assumption about the value function, this proposition tells us that each agent in subnetwork formed by group A (the network using the Myerson value as an allocation rule), will have an equal number of connections. Furthermore, under this condition the normalized Myerson value "looks" like the egalitarian rule. This result can be generalized for any allocation rule Y^A, provided $Y^B = Y^e$. Whenever $m = 1$, the probability is linear in group size and equal to the group size divided by the total number of agents. This forces the expected allocation to be constant (and equal to $\frac{1}{N}$) for every agent in group B. For any equilibrium, the expected allocation for group A must be greater than or equal to $\frac{1}{N}$. Given the linear probability, the allocation for group A must be bigger than or equal to $\frac{1}{|A|}$. This can only occur when each agent is allocated $\frac{1}{|A|}$.

The following two results examine the implications of the existence of subnetwork in which individuals are allocated an equal share of the prize under the Myerson value. First, the size for network A is bounded above. Secondly, if the network size for A is equal to the upper bound, then an equilibrium implies the individual allocations must be equivalent to the egalitarian rule.

Lemma 1. *Let g be an arbitrary network and suppose $m < 1$ in the CSF. If $Y_i^A = Y_j^A$ for all $i, j \in A$, then $|A| \leq |B| + 1$ in any equilibrium partition.*

Proof. I must have $\sum_{i \in A} Y_i = v(g|A)$, therefore if each $Y_i = Y^*$ is equal then $\sum_{i \in A} Y_i = |A|Y^* = v(g|A)$. Therefore, the normalized allocation is equal to $\frac{1}{|A|}$. Therefore, for each $i \in A$:

$$\frac{1}{|A|}\frac{|A|^m}{|A|^m + |B|^m} \geq \frac{1}{|B|+1}\frac{(|B|+1)^m}{(|A|-1)^m + (|B|+1)^m} \tag{2.20}$$

This can be rewritten as:

$$(|B|+1)^{1-m}((|A|-1)^m) + |B| + 1 \geq |A| + |B|^m|A|^{1-m} \tag{2.21}$$

Assume $|A| > |B| + 1$, then $((|A|-1)^m)(|B|+1)^{1-m} > |A|^{1-m}|B|^m$, which may be rewritten as $((|A|-1)^m)|A|^{m-1} > ((|B|+1)^{m-1})|B|^m$. This cannot be satisfied if $m < 1$. $\square$

Proposition 7. *Let g be an arbitrary network and suppose $m < 1$ in the CSF. The only equilibrium partition with $|A| = |B|+1$ must have $Y_i^A(g|_A, v) = \frac{1}{|A|}$. Furthermore, no equilibrium partition with $|A| > |B| + 1$ exists.*

Proof. If I assume $|A| = |B| + 1$, the following must be true from (2.5):

$$Y_i(g|_A, v) \geq \tfrac{1}{|A|}$$

This only holds for all $i \in A$, $Y_i(g|_A, v) = \frac{1}{|A|}$. Now suppose $|B| + 1 < |A|$, then the following two facts are true due to the curvature of the contest success function for $m < 1$:

$$\frac{|A|}{|A|+|B|} > \frac{|A|^m}{|A|^m+|B|^m} \tag{2.22}$$

$$\frac{(|B|+1)^m}{(|B|+1)^m+(|A|-1)^m} > \frac{|B|+1}{|A|+|B|} \tag{2.23}$$

Combining these with (2.5) and simplifying:

$$\tfrac{k_i}{\sum_{j\in A} k_j} > \tfrac{1}{|A|}$$

This must hold for each $i \in A$, which is a contraction since the total allocation must sum to one. $\square$

The following results characterize two types of equilibria for the general case. In the first case, each individual in the subnetwork formed by group A is offered an equal share of the prize. However, these types of equilibria are not guaranteed to exist. In Proposition 9, I characterize an equilibria which always exists and in doing so, prove existence for all ranges of the parameter m in the CSF.

Proposition 8. *Let g be an arbitrary network and suppose $m < 1$ in the CSF. If each agent i in the subnetwork formed by group A has an equal number of connections and $\frac{N-1}{2} \leq |A| \leq \frac{N+1}{2}$, then it forms an equilibrium partition with $g|B$ if there does not exist $j \in B$ such that $k_j^{A\cup\{j\}} > k_A$ where k_j is the degree of agent j in $g|_{A\cup\{j\}}$. If $|A| < \frac{N-1}{2}$, $g|A$ and $g|B$ form an equilibrium partition only if $\frac{|A|}{|A|-1}k_A > k_j^{A\cup\{j\}}\ \forall\, j \in B$.*

Proof. For Ineq. (2.5) to be satisfied, the group sizes must satisfy $|A| \leq |B|+1$ by the previous Lemma. For Ineq. (2.6), one must have:

$$\frac{1}{B}\frac{|B|^m}{|A|^m+|B|^m} \geq \frac{k_j^{A\cup\{j\}}}{\displaystyle\sum_{i\in A} k_i^A + 2k_j^{A\cup\{j\}}}\frac{(|A|+1)^m}{(|A|+1)^m+(|B|-1)^m} \tag{2.24}$$

If $|A|+1 \geq |B|$, then

$$\frac{1}{B} \geq \frac{k_j^{A\cup\{j\}}}{\sum_{i\in A} k_i^A + 2k_j^{A\cup\{j\}}} \tag{2.25}$$

this can be rewritten as follows:

$$\frac{\displaystyle\sum_{i\in A} k_i^A}{|B|-2} \geq k_j^{A\cup\{j\}}$$

Since $k_i^A = k_A \; \forall \; i \in A$,

$$k_A \geq \frac{|A|k_A}{|B|-2} \geq k_j^{A\cup\{j\}}$$

Therefore, provided each agent $j \in B$ has a number of connections to agents in A that is less than or equal to the number of connections each agent in A has, $I2$ will be satisfied. If $g|_A$ forms a clique, then $k_A = |A|-1$. The above inequality is always true since the maximum number of connections an agent might have is $|A|$. Thus, whenever $|B|-1 \leq |A| \leq |B|+1$ a clique of size $|A|$ is always part of an equilibrium partition. This translates to $\frac{N-1}{2} \leq |A| \leq \frac{N+2}{2}$.

For the last statement, if $|A|+1 < |B|$, the following two facts are true due to the curvature of the contest function for $m < 1$:

$$\frac{B}{A+B} > \frac{B^m}{A^m+B^m} \tag{2.26}$$

$$\frac{(A+1)^m}{(A+1)^m + (B-1)^m} > \frac{A+1}{A+B} \tag{2.27}$$

Combining these with (2.6) and simplifying, the last statement of the proposition is obtained.

$$\tfrac{|A|}{|A|-1} k_A > k_j^A$$

□

Corollary 2. *If $m < 1$ in the CSF, a clique of size of $\frac{N-1}{2} \leq |A| \leq \frac{N+1}{2}$ is always part of an equilibrium partition. If $|A| < \frac{N-1}{2}$, it is only part of an equilibrium partition if there does not exist $j \in B$ such that $g|_{A \cup \{j\}}$ is a clique of size $|A| + 1$.*

Under this type of equilibrium, the agents in group A each receive the same allocation. The allocation rule "looks" like the egalitarian rule. An example of this type of equilibrium is a clique. Finding a clique of a certain size is a well-known problem in graph theory and computer science and is known to NP-Complete. Therefore, it is unlikely[1] there is an efficient way to enumerate this type of equilibria for an arbitrary graph. This result generalizes for value functions which assign different values to each link. Provided group B uses the egalitarian allocation rule, allocation rule Y^A must assign an equal share of the prize to each agent in any stable subnetwork $g|_A$. Each agent must contribute an equal amount to the value of the subnetwork.

Proposition 9. *Suppose $m < 1$ in the CSF. The densest subnetwork of size $|A| < \frac{N+1}{3}$ and any subnetwork $g|B$ forms an equilibrum partition.*

---

[1] Provided $P \neq NP$

Proof. Suppose $g|A$ is the densest subgraph of size $|A| < \frac{N+1}{3}$. Since A is the densest subgraph of size $|A| < \frac{N+1}{3}$, then the following is true for each $j \in B$:

$$\frac{\sum_{i \in A} k_i}{2|A|} \geq \frac{\sum_{i \in A} k_i + 2k_j^{A \cup \{j\}}}{2(|A|+1)} \tag{2.28}$$

Here $k_j^{A \cup \{j\}}$ denotes the degree of agent j in the subgraph $g|(A \cup \{j\})$. This simplifies to the following:

$$\frac{\sum_{i \in A} k_i}{2|A|} \geq k_j^{A \cup \{j\}} \tag{2.29}$$

I obtain the following from the equilibrium inequality for $j \in B$.

If $|A| + 1 \leq |B|$, then following holds due to the curvature of the contest success function:

$$\frac{|B|}{|A| + |B|} > \frac{|B|^m}{|A|^m + |B|^m} \tag{2.30}$$

$$\frac{(|A|+1)^m}{(|A|+1)^m + (|B|-1)^m} > \frac{|A|+1}{|A|+|B|} \tag{2.31}$$

These two relationships imply the following:

$$\frac{\sum_{i \in A} k_i}{2|A|} \geq \frac{|A|-1}{2|A|} k_j^{A \cup \{j\}} \tag{2.32}$$

For the inequality involving agents in network A, whenever $|A| \leq |B|$, the following bounds hold:

$$\frac{1}{2} \geq \frac{|A|^m}{|A|^m + |B|^m} \tag{2.33}$$

$$\frac{(|B|+1)^m}{(|B|+1)^m+(|A|-1)^m} \geq \frac{1}{2} \tag{2.34}$$

Therefore, if the following holds Ineq. (2.5) will be satisisfied.

$$\frac{k_i}{\sum_{j\in A} k_j} \geq \frac{1}{|B|+1} \tag{2.35}$$

Rearranging and dividing both sides by $2|A|$, the above equation is now written as:

$$\frac{|B|+1}{2|A|} k_i \geq \frac{\sum_{j\in A} k_j}{2|A|} \tag{2.36}$$

Furthermore, since $|A|$ is the densest subgraph, each agent's degree must be bigger than or equal to the density. Otherwise, removing the agent with the smallest degree increases the density which would contradict the fact that $|A|$ is the densest subgraph with an upper bound. Therefore, Ineq. (2.36) is satisfied if $|B|+1 \geq 2|A|$. This corresponds to $|A| \leq \frac{N+1}{3}$. $\square$

The densest subgraph with a upper bound on the size can always be found, since there are finite number of subgraphs. Finding the densest subgraph of an exact size is another NP-Complete problem, as is finding the densest subgraph with a lower bound on the size. The complexity of finding the densest subgraph with an upper size bound is unknown. This proposition generalizes for value functions which assign different values to each link, with a slightly modified definition of density that takes the value of each edge into account.

3. Implications for the outcome of the contest

I have proven existence of an equilibrium for all ranges of the parameter m. If $m > 1$, the only equilibria involve every agent joining one group or the other. If $m = 1$, in any equilibrium, the network using the Myerson value (network A) must allocate an equal share of the prize to each agent. Consequently, the maximum winning probability for this network corresponds to the largest subgraph where each vertex has an equal number of connections. In the case $m < 1$, the size of the network using the Myerson value is bounded by above $\frac{N+1}{2}$. Whereas the size for the network using the egalitarian rule (network B) is not necessarily bounded above by any number. Furthermore, if $|A| = |B| + 1$ then group A must allocate each individual an equal share of the prize. Thus, the winning probability can only be greater than one half when each agent receives an equal share of the prize and this outcome corresponds to the existence of a subgraph of a certain size in which each vertex has an equal number of connections. This tells us that the two probabilities cannot diverge in favor of network A, whereas network B may have a much larger probability of winning the contest for some underlying graphs than network A.

D. Extension of the Model

In the previous section, the outcome of the contest between the two competing networks did not depend on any property of the network, other than the size. It seems reasonable, that if agents must communicate during the contest, that the outcome of the contest must depend on how the agents are able to communicate. One seemingly reasonable assumption is that networks with more connections are able to communicate and share information more efficiently than those with fewer connections. This is captured by the following contest success function:

$$p_A(g|A) = \frac{(\sum_{ij \in g|A} v\{ij\})^m}{(\sum_{ij \in g|A} v\{ij\})^m + (\sum_{ij \in g|B} v\{ij\})^m} \tag{2.37}$$

or simply,

$$p_A(g|A) = \frac{v(g|A)^m}{v(g|A)^m + v(g|B)^m} \tag{2.38}$$

where $v(g|A) = \sum_{ij \in g|A} v\{ij\}$. Here $p(g|_A)$ is the winning probability for the network formed by the subgraph $g|_A$.

The equilibrium notion is still based on expected allocation (payoff) and the two equilibrium defining inequalities become the following:

For $i \in A$,

$$Y_A^i(g|A, v)\frac{v(g|A)^m}{v(g|A)^m + v(g|B)^m} \geq Y_B^i(g|B \cup i, v)\frac{v(g|B \cup \{i\})^m}{v(g|A \backslash \{i\})^m + v(g|B \cup \{i\})^m} \tag{2.39}$$

For $j \in B$,

$$Y_B^j(g|B, v)\frac{v(g|B)^m}{v(g|A)^m + v(g|B)^m} \geq Y_A^j(g|A \cup j, v)\frac{v(g|A \cup \{i\})^m}{v(g|B \backslash \{i\})^m + v(g|A \cup \{i\})^m} \tag{2.40}$$

Following the standard of value function over graphs, the value of a single node is zero. Therefore, within this framework, the grand alliance is always an equilibrium partition. If both subnetworks contain zero connections or zero value, the contest function is undefined. To avoid this, I will asume that the underlying network is not the empty network and that there always exists a partion of the graph such that at least one subnetwork has non-zero value.

1. Example

Suppose $v\{ij\}$ is very large and for all $lk \in g$ such that $lk \neq ij$, $v\{lk\} = 1$. Furthermore, let $Y_A = Y^{MV}$ and $Y_B = Y^e$ as before. Consider the two subgraphs formed by $A = \{i, j\}$ and $B = N \backslash A$. The first equilibrium inequality for the example has the following form:

For both $i, j \in A$

$$\frac{1}{2} \frac{v\{ij\}^m}{v\{ij\}^m + v(g|B)^m} \geq \frac{1}{N-1} \tag{2.41}$$

This simplifies to the below inequality.

$$\frac{N-3}{2} v\{ij\}^m \geq v(g|B)^m \tag{2.42}$$

The second equilibrium inequality for this example is given below.

For $k \in B$

$$\frac{1}{N-2} \frac{v(g|B)^m}{v\{ij\}^m + v(g|B)^m} \geq \frac{2}{v\{ij\} + 2} \frac{v(g|A \cup \{k\})^m}{v(g|B \backslash \{k\})^m + v(g|A \cup \{k\})^m} \tag{2.43}$$

Without loss of generality one may assume that each agent k is connected to both agents i and j, since the inequality will hold if the agent is connected to one or none of agents i, j. I know that $\frac{v(g|A \cup \{i\})^m}{v(g|B \backslash \{i\})^m + v(g|A \cup \{i\})^m} \leq 1$. Therefore, if the following holds, (2.43) will also be satisfied.

$$\frac{1}{N-2}\frac{v(g|B)^m}{v\{ij\}^m + v(g|B)^m} > \frac{2}{v\{ij\}+2} \tag{2.44}$$

which yields the following:

$$v\{ij\} - \frac{v\{ij\}^m}{v(g|B)^m} > \frac{N-6}{2} \tag{2.45}$$

By assumption, there is no upper bound on $v\{ij\}$. Therefore, one may make it as large as one likes without affecting the stability relationships. Clearly, $\lim v\{ij\} \to \infty$ will make $\frac{v\{ij\}^m}{v\{ij\}^m + v(g|B)^m} \to 1$ since $v(g|B)$ is fixed. Thus, there exist equilibria in which the winning probability for the group using the Myerson value as an allocation rule are arbitrarily close to one. This stands in contrast to the previous results where the winning probability for the same group had an upper bound for all non-trivial equilibria.

2. Symmetric case

If the same assumption of symmetry on the value of connections from the previous model is imposed, the CSF becomes the following:

$$p_A(g|A) = \frac{(\sum_{i\in A} k_i)^m}{(\sum_{i\in A} k_i)^m + (\sum_{j\in B} k_j)^m} \tag{2.46}$$

Our two equilibrium inequalities become the following:

I1) For $i \in A$

$$\frac{k_i}{\sum\limits_{l \in A} k_l} \frac{(\sum\limits_{l \in A} k_l)^m}{(\sum\limits_{l \in A} k_l)^m + (\sum\limits_{j \in B} k_j)^m} \geq \frac{1}{|B|+1} \frac{(\sum\limits_{j \in B \cup \{i\}} k_j)^m}{(\sum\limits_{l \in A \setminus \{i\}} k_l)^m + (\sum\limits_{j \in B \cup \{i\}} k_j)^m} \tag{2.47}$$

I2) For $j \in B$

$$\frac{1}{|B|} \frac{(\sum\limits_{j \in B} k_i)^m}{(\sum\limits_{i \in A} k_i)^m + (\sum\limits_{j \in B} k_j)^m} \geq \frac{k_j^A}{\sum\limits_{l \in A \cup \{j\}} k_l} \frac{(\sum\limits_{l \in A \cup \{j\}} k_j)^m}{(\sum\limits_{l \in A \cup \{j\}} k_l)^m + (\sum\limits_{j \in B \setminus \{j\}} k_j)^m} \tag{2.48}$$

3. A special case: The complete network

Again, the complete network must be considered as a special case. Under the complete network each subnetwork of size n is a clique with $\frac{n(n-1)}{2}$ connections. The contest success function becomes the following:

$$p_A(g|A) = \frac{(|A|^2 - |A|)^m}{(|A|^2 - |A|)^m + (|B|^2 - |B|)^m} \tag{2.49}$$

The following figure shows the contest success function for group A with $N = 100$. Whenever $m = .5$, the contest success function is approximately linear in the size of the network, due to the additional term in the CSF. Using the same parameter value for m, this contest success function is more sensitive to group size as compared to the previous model. This illustrated in Fig. 5.

Proposition 10. *Suppose g is the complete graph, N is even, and the following holds:*

$$m \leq \frac{log(N/2-1) - log(N/2+1)}{log(N/2-1) + log(N/2-2) - log(N/2+1) - log(N/2)}$$

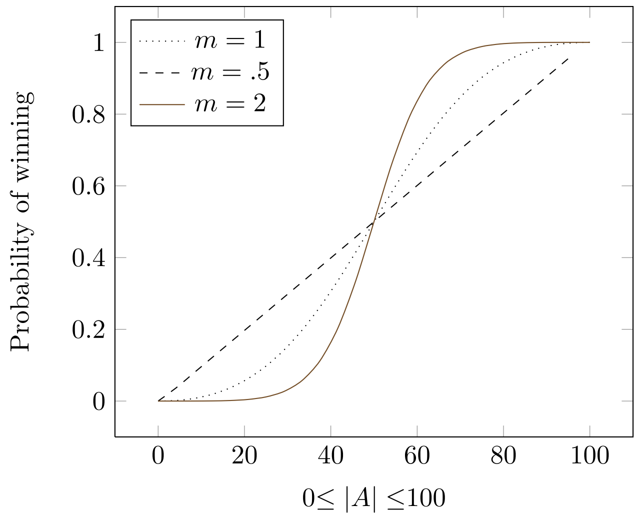

Fig. 5. CSF with $N = 100$

Then any graph partition such that $|A| = |B|$ is an equilibrium.

Proof. Suppose $|A| = |B|$, the inequalities are identical and given by

$$\frac{1}{2A} \geq \frac{1}{|B|+1} \frac{|B+1|^m |B|^m}{|B+1|^m |B|^m + |A-1|^m |A-2|^m} \tag{2.50}$$

If N is even, $|A| = \frac{N}{2}$ and the above reduces to:

$$\frac{1}{N} \geq \frac{(N/2+1)^{m-1}(N/2)^m}{(N/2+1)^m (N/2)^m + (N/2-1)^m (N/2-2)^m} \tag{2.51}$$

This inequality can be rewritten as follows:

$$(N/2-1)^{m-1}(N/2-2)^m \geq (N/2+1)^{m-1)}(N/2)^m \tag{2.52}$$

Taking the logarithm of both sides and rearranging terms, yields the statement of the proposition.

$$m \leq \frac{log(N/2-1) - log(N/2+1)}{log(N/2-1) + log(N/2-2) - log(N/2+1) - log(N/2)} \tag{2.53}$$

□

The behavior of these equilibria as N grows large can be seen by looking at the behavior of the upper bound on m. As N grows large, the upper bound approaches one half. This makes intuitive sense because the number of connections in each component is $(N/2)(N/2-1)$ which behaves approximately like N^2 for large values of N.

Proposition 11. *Suppose g is the complete graph, N is odd, then there exist values of m such that subnetworks with $|B|-1 \leq |A| \leq |B|+1$ form an equilibrium partition.*

Proof. Whenever $|A| = |B| - 1$, the second inequality is satisisfied trivially. Similarly, if $|A| = |B| + 1$ the first inequality is satisisfied. Therefore, in either case only one inequality depends on the parameter m.

Consider the case where $|A| = |B|-1$. The first inequality becomes the following:

$$\frac{|A|^{m-1}(|A|-1)^m}{|A|^m(|A|-1)^m + (|A|+1)^m|A|^m} \geq \frac{(|A|+2)^{m-1}(|A|+1)^m}{(|A|+2)^m(|A|+1)^m + (|A|-1)^m(|A|-2)^m} \tag{2.54}$$

Unlike the previous case, the upper bound for m has no closed form. However, the solution to the transcendental equation can be solved approximately using numerical methods. As before, the upper bound depends on N. Furthermore, this upper bound is different from the upper bound obtained in the previous case. Fig. 6 shows the approximate values of m such that the LHS and RHS of the above inequality are equal. Any value of m smaller than this critical value, result in an equilibrium such that $|A| = |B|-1$. These critical values can be found for the case in which $|A|+1 = |B|$ in the same manner. $\square$

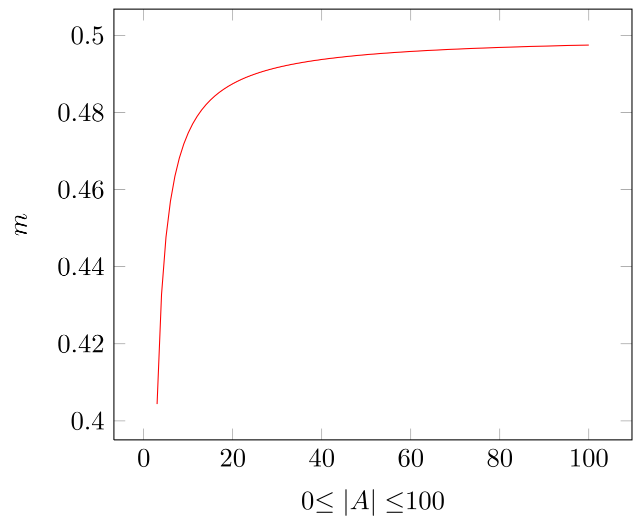

Fig. 6. Critical values of m as a function of $|A|$.

The differences between the two models is nicely illustrated in this special case. As illustrated above, allowing the CSF to depend on the number of connections makes the equilibria more "sensitive" to size of the networks

4. The general case

To contrast the two models, I will examine what, if any, additional conditions must be placed on the each subnetwork so that the equilibria discussed in section remain equilibria.

Consider the case in which $m = 1$. Proposition 6 showed that any partition such that each agent in the network using the Myerson value must have an equal number of connections and hence an equal share of the prize. The following proposition will provide contrast to this result. As in Proposition 6, it is assumed that g is an arbitrary network, and that $m = 1$ in the contest success function, however, the CSF used is defined by equation (16).

Proposition 12. *In order for clique of size $|A|$, and a subnetwork $g|B$ with $|A|-1 \geq |B|$, to form an equilibrium partition, the following condition is sufficient: For each $j \in B$*

$$\frac{|A|(|A|-1)}{|B|-1} > k_j^A \tag{2.55}$$

where k_j^A denotes the number of connections agent j has in $g|_{A\cup\{j\}}$.

Proof. Since the each agent in A has an equal number of connections, the inequality becomes the following:

$$|A|-1 \geq \frac{\frac{1}{2}\sum_{j\in B} k_j + 2k_i^B}{|B|+1} \frac{|A|(|A|-1)+\frac{1}{2}\sum_{j\in B} k_j}{(|A|-1)(|A|-2)+\frac{1}{2}\sum_{j\in B} k_j + 2k_i^B} \tag{2.56}$$

Notice that since $|A|-1 > |B|$,

$$\frac{|A|(|A|-1)+\frac{1}{2}\sum_{j\in B} k_j}{(|A|-1)(|A|-2)+\frac{1}{2}(\sum_{j\in B} k_j + 2k_i^B)} > 1 \tag{2.57}$$

This is true because the maximum degree of agent i in $g|_{B\cup\{i\}}$ is $|B|$. Therefore, the following must hold for each $i \in A$:

$$|A|-1 \geq \frac{\sum_{j\in B} k_j + 2k_i^B}{2(|B|+1)} \tag{2.58}$$

For each agent i, the density of the subgraph $g|_{B\cup\{i\}}$ must be less than or equal to $|A|-1$. Given that $|A|-1 > |B|$, this is always satisfied. The maximum density of such a network is $\frac{|B|(|B|+1)}{|B|+1} = |B|$. Therefore, (2.39) is always satisfied.

Ineq. (2.40) becomes:

$$\frac{1}{|B|}\frac{\frac{1}{2}\sum_{l\in B}k_l}{|A|(|A|-1)+\frac{1}{2}\sum_{l\in B}k_l} \geq \frac{k_j^A}{\sum_{i\in A}k_i+2k_j^A}\frac{\sum_{i\in A}k_i+2k_j^A}{\sum_{i\in A}k_i+2k_j^A+\sum_{l\in B}k_l-2k_j} \tag{2.59}$$

Since $|A|-1>|B|$, the winning probablity for network B must be less than one half. Likewise, the winning probability for network A with one additional agent is greater than one half. Therefore,

$$\frac{1}{|B|} > \frac{k_j^A}{\sum_{i\in A}k_i+2k_j^A} \tag{2.60}$$

Rearranging terms, the statement in the proposition is obtained.

$$\frac{|A|(|A|-1)}{|B|-1} > k_j^A \tag{2.61}$$

□

The number of connections divided by the size of network B less one must be greater than any agent j's connections in $g|_{A\cup\{j\}}$. In other words, all agents in B must have more connections with other agents in network B than potential connections in network A. Previously, the connections between groups did not play a role for the group using the egalitarian rule in these equilibria.

Fig. 7 shows how the equilibria in Proposition 6 may not remain equilibria under the new CSF when $m=1$. The agent in group B with four potential connections to the agents in group A may increase their expected allocation from 1/12 to 8/63 by defecting. However, if the CSF only depends on the size of the networks, this

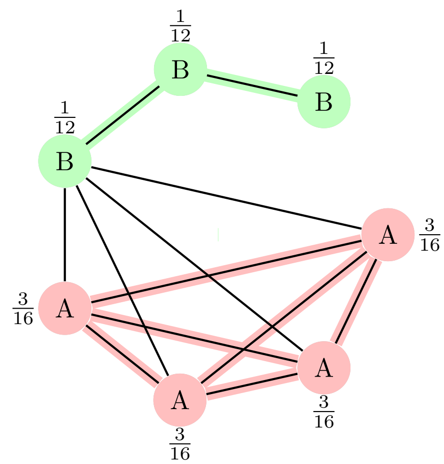

Fig. 7. Partition which does not form an equilibrium

same partition forms an equilibrium. This example shows that additional conditions may need to placed on the partitions which form equilbria when the contest success function depends only on the size of the subnetworks, in order for these partitions to remain equilibria under the new CSF.

E. Conclusion

I have shown that equilibrium partitions always exist for when two competing networks offer the two allocation rules discussed. In order for both partitions to be non-empty, the parameter m in the CSF must be less than or equal to one. That is, the probability of winning the contest must not be too sensitive to the network size. The winning probability for the group using the egalitarian rule can be very close to one. In contrast, the winning probability for the group using the Myerson value cannot be much larger than one half. Furthermore, in some cases the resulting stable partitions will allocate each individual equal shares of the potential prize, despite the fact that the allocation rules are different. The presence of the egalitarian rule keeps the other group “honest”, in the sense that the allocation cannot be too far from an equal split of the prize. Modifying the contest sucess function so that each network’s winning probability depends on the number of connections within each network, allows equilibria in which the Myerson value is the dominant allocation rule, in the sense that the winning probability may be very close to one. Future work may include allowing for directed networks and hierarchical allocation rules or generalizing the results for arbitrary allocation rules.

CHAPTER III

DYNAMIC NETWORK FORMATION WITH REINFORCEMENT LEARNING

A. Introduction

In many network formation models, such as [29], [13], [15], and [1], the assumption of complete information plays a key role in determining what types of networks will form. The interested reader may refer to [10] or [14] for surveys on models of network formation. In [29], the network formation process is deterministic with agents myopically modifying their links to improve their current payoff. Using the payoff structure and stability notion defined in the Connections Model proposed by [9], Watts finds that, depending on parameter values, this network formation will always converge to "stable" network. [13] extend this model to a stochastic setting by allowing the possibility that agents choose the wrong action with some small probability. As the probability of making a mistake approaches zero, two scenarios arise: the network formation process converges to a single network or forms a closed cycle of networks. In each model, agents know the current state of the network, as well as how each action would affect their payoff, and make decisions based on this knowledge. [1] proposed a game-theoretic model of network formation where agents possess complete information about the network structure and choose their link-formation strategies based on this information. The authors examine what types of networks may emerge and show that for many different parameter values, the predicted equilibrium network is a star-network even though agents are indistinguishable with respect to payoffs and costs. It is unlikely that the structure of any economic network would be known to all those involved. It is easy to imagine a scenario that one would have limited information regarding network structure, for example, privacy settings on Facebook may prohibit an individual from viewing the friends of a friend. If the assumption of complete information is removed, one might be able to understand how information drives network formation.

Another contribution of [1] was the introduction of learning dynamics to the study of network formation. The learning dynamics examined were based on myopic best responses where agents exhibit some inertia when choosing their strategies. The assumptions made for the learning process may not be realistic for large networks, since each agent must observe the strategies of all other agents. On the other-hand, reinforcement learning allows the agents to merely observe their own payoff and adjust the probabilities associated with each action, which seems more plausible for situations where observing all agents who make up the network is not realistic. Q-learning, first proposed by [28], is a reinforcement learning algorithm which allows for state-dependent learning, and therefore will allow agents to observe some information about the network. Network formation with reinforcement learners was examined by [24]. However, in their model, the payoffs were independent of network structure. In this paper, I will explicitly capture the trade-off between the cost of maintaining direct links to others and the informational benefit of being well connected by using the Connections Model presented by [9].

My approach is to model network formation as a dynamic process driven by reinforcement learning. See [4] for a comprehensive survey of multi-agent reinforcement learning in non-network contexts. Agents meet over time and choose to maintain or delete existing connections, or add a new connection or maintain non-existent connections. These actions are chosen according a probability distribution and these probabilities adjust over time based on experience. This experience changes in response to the payoffs received after choosing a particular action. It may be helpful to imagine the network which evolves over time as a communication network in which a link between two agents indicates they have chosen to share information. After the active agents have chosen their actions, the communication network changes accordingly. As time passes, agents choose the action which has yielded the highest payoff increasingly often. I examine three informational settings in this paper. I first examine what happens when agents are unaware of who they are attempting to establish social ties with, the structure of the network, and how a certain network structure would benefit them. They merely observe their current and past payoffs. In the second scenario, agents observe the individuals they are attempting to form connections with, as well as, the payoff received each period. A third informational setting is examined in which agents observe the number of connections they have.

In each of the models described above, I first characterize which networks may be absorbing networks for the process and then further examine the convergence and stability properties of these absorbing networks using simulations. Each of the models are simulated and the long-run behavior is discussed. My results suggest that certain networks, which "pairwise stable" in the Connections Model, are not absorbing networks if agents do not observe any information beyond their payoff. However, if one increases the amount of information available to agents and reformulates the learning rule to be state dependent, these pairwise stable networks become absorbing networks for the network formation process.

This paper contributes to the theory of network formation in two ways . First, reinforcement learning is introduced to a model of network formation in which the agents' payoffs depend explicitly on the network structure. The second contribution is the use of state-dependent reinforcement learning to allow the agents access to limited information during the network formation process.

The rest of this paper is organized as follows. Section B presents some preliminary definitions for networks. Section C defines the utility function specified in the

Connections Model and defines pairwise stability and efficiency for networks. Section D introduces the basic model and Section E allows for limited information by incorporating a state-dependent learning rule. Section 6 presents the results of simulations carried out in order to examine the convergence properties of each model as well as the effect the various parameters have on the long-run behavior of each model.

B. Representing Networks

Before each model is introduced, some preliminary definitions and notation are needed. This notation follows the conventions established in the previous economic literature on networks and network formation whenever possible.

1. Nodes and agents

A set $N = \{1, 2, \ldots, n\}$ is the set of nodes belonging to a network. Nodes can represent individuals, firms, countries, or objects such as webpages and may be referred to as "nodes", "agents", "individuals", or "players".

2. Graphs and networks

The traditional representation of a network is an undirected labeled graph. Nodes are either connected or not. A directed graph can be used to represent networks where one-sided relationships may be present. Trading relationships or friendship networks are usually represented by undirected graphs, as both sides must establish and maintain the relationship. The formal model of networks is given below:

A network (N, g) consists of a set of nodes $N = \{1, 2, \ldots, n\}$ and g, a set of links, or edges, present in the graph. Links may also be called called edges or connections. For instance, $g = \{\{1, 2\}, \{2, 3\}\}$ or simply, $g = \{\{12\}, \{23\}$ indicates links between

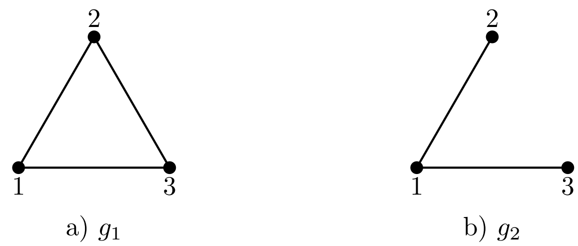

Fig. 8. The representations for the networks $g_1 = \{12, 13, 23\}$ and $g_2 = \{12, 13\}$.

nodes 1 and 2, as well as nodes 2 and 3. The notation ij will be used to represent the edge connecting node i and j. An example of this notation is shown in Fig. 8. Furthermore, I will abuse notation at times and write $ij \in g$ if nodes i and j are connected in network (N, g). I will take $g + ij$ to mean the network obtained by adding edge ij to the set of connections of g. More precisely, $g + ij = g \cup \{ij\}$. Similarly, $g - ij$ will represent the network obtained by deleting edge ij from the set of connections of g or mathematically, $g - ij = g \setminus \{ij\}$.

C. The Connections Model

The Connections Model was first proposed by [9]. In the model, links between nodes or agents, represent social relationships that present both benefits to the agents and a require cost in order to maintain the link. Agents also benefit from their indirect connections or the "friends of a friend", so that there is a trade-off between maintaining direct links and the benefits received from the overall connectivity of the network. Formally, the model is defined as follows: The utility or payoff agent $i \in \{1, 2, 3 \ldots, n\}$ receives from network g is

$$u_i(g) = w_i + \sum_{j \neq i} \delta_{ij}^{d_{ij}(g)} - k_i(g) c_{ij} \tag{3.1}$$

where $d_{ij}(g)$ is the length of the shortest path distance between nodes i and j, δ_{ij} represents the value received by i from j, $k_i(g)$ is the agent i's degree in the network, c_{ij} is the cost of maintaining the link between i and j, and w_i is constant. Furthermore, it is assumed that $\delta_{ij} \in (0,1)$ so that indirect links are of less value than direct links. If there is no path connecting nodes i and j, then $d_{ij}(g) = \infty$. The Symmetric Connections Model assumes $\forall\ i, j \in N \quad \delta_{ij} = \delta$, $w_i = w_j$, and $c_{ij} = c$. I will assume the individual payoffs are symmetric throughout the rest of this paper.

1. Stability and efficiency of social networks

In order to characterize the networks that arise when formation of links requires consent of both agents involved, [9] introduced a notion of network stability which they called pairwise stability. As implied by the definition, no one individual wishes to make any changes to the connections in the network.

Definition 11. *A network g is* pairwise stable *if for each i, j the following hold:*

i) $ij \in g, u_i(g) \geq u_i(g - ij)$ *and* $u_j(g) \geq u_j(g - ij)$

ii) $ij \notin g$, *if* $u_i(g) < u_i(g + ij)$, *then* $u_j(g) > u_j(g + ij)$

This notion of stability is independent of the the network formation process and therefore, provides a good benchmark for my model. Although the concept of pairwise stability is somewhat weak, it makes some strong predictions for the stability of networks. This is illustrated in the following theorem which addresses the existence and uniqueness of pairwise stable networks.

Theorem 1. *[9] In the symmetric connections model, a pairwise stable network exists for all N and is given by:*

i) The unique pairwise stable network is the complete graph, g^N if $c < \delta - \delta^2$, .

ii) A star[1] *encompassing all players if* $\delta > c$ *and* $\delta - c \leq \delta^2$*,*

iii) The empty network (no links) if $\delta \leq c$

In the parameter range $\delta < c$, the notion of pairwise stability rules out any network in which an agent possesses one link. In particular, the star network cannot be stable. Another property of networks one may be concerned with is efficiency. Jackson and Wolinsky use the following definition for efficiency:

Definition 12. *A network is* g efficient *if* $\sum_i u_i(g) \geq \sum_i u_i(g')$ *for all* $g' \in \mathcal{G}$

Here $\mathcal{G}$ denotes the set of all networks of size n. Here the concept of efficiency is equivalent to the idea of social welfare for a given network structure. For the following parameter ranges, the efficient networks were shown to be unique and are characterized as follows:

Theorem 2. *[9] In the symmetric connections model, a unique efficient network exist for all* n *and is given by:*

i) the complete graph g^N *if* $c < \delta - \delta^2$.

ii) a star encompassing everyone if $\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$

iii) the empty network if $\delta + \frac{n-2}{2}\delta^2 < c$

These results on stability and efficiency have used as a benchmark for the models presented in Watts [29] and Jackson [13] and will be used as a comparison for the models presented in the next two sections.

[1] A star network consists of one central agent and periphery agents, each connected only to the central agent.

D. The RL Model

Agents meet over time and decide whether or not to form connections or sever existing connections. Time is represented by a countably infinite set $T = \{1, 2, 3, \ldots\}$. The network structure at time t is represented by an undirected graph which is denoted as g_t. The initial network g_0 is a parameter of the model. The network evolves slowly enough so that only two agents may alter their shared connection at a given point in time. These two agents meet randomly with a probability of $\frac{1}{\binom{n}{2}} = \frac{2}{n(n-1)}$. Once agent i and agent j have met, they choose actions a_i and a_j from the set $\mathcal{A} = \{add, delete\}$ according to distributions $(p_i(t), 1 - p_i(t))$ and $(p_j(t), 1 - p_j(t))$, respectively, which are defined below. It is assumed that agents must both agree to add or maintain an existing connection, however, individuals may unilaterally sever a connection. Therefore, if both agent i and agent j choose to add a link and $ij \notin g_t$, then $g_{t+1} = g_t + ij$. If $ij \in g_t$, then $g_{t+1} = g_t$. Likewise, if either agent i or agent j chooses to delete link ij and $ij \in g_t$ then $g_{t+1} = g_t - ij$. If $ij \notin g_t$ and one agent chooses not to add the link, then $g_{t+1} = g_t$.

Let $Q_0^i = [Q_0^i(a), Q_0^i(d)]$ denote the initial assessments agent i has for each action. After the agents have been randomly matched, they choose their actions probabilistically based on their current assessments. The probability that agent i chooses to add a connection is denoted by $p_i(t)$ and the probability that agent i chooses to delete a connection is given by $1 - p_i(t)$. Throughtout the rest of the paper I will denote $p_i(t)$ by $p_{i,t}$ for compactness. This probability is defined by:

$$p_{i,t} = \frac{e^{Q_t^i(a)/\tau_t^i}}{e^{Q_t^i(a)/\tau_t^i} + e^{Q_t^i(d)/\tau_t^i}} \tag{3.2}$$

This choice rule is known by various names including logistic choice or softmax selection. Here τ_t^i is sometimes called the temperature parameter[2] or rationality parameter. If τ_t^i is very large, all actions are chosen with approximately equal probabilities. If τ_t^i is very close to zero, the action with the highest payoff assessment is chosen with probability close to one. I will assume that for each agent i, $\tau_t^i \to 0$ as $t \to \infty$, so that, as time increases, each agent i chooses the action with highest Q-value more and more often. One typical temperature function is given by $\tau(t) = \eta^t$ where $\eta < 1$.

Since agents may not unilaterally establish a connection with another agent, the probability that $ij \in g_{t+1}$ is $p_{i,t}p_{j,t}$. After choosing their actions at time t, the network transitions according to the following probabilities:

$$g_{t+1} = \begin{cases} g_t + ij \text{ with probability } p_{i,t}p_{j,t} \\ g_t - ij \text{ with probability } 1 - p_{i,t}p_{j,t} \end{cases}$$

Agents then receive a payoff, which they are able to observe. The payoff received by agent i given agent i chooses to add a connection is:

$$\pi_{i,t+1}(a) = \begin{cases} u_i(g_t + ij) \text{ with prob } p_{j,t} \\ u_i(g_t - ij) \text{ with prob } 1 - p_{j,t} \end{cases}$$

If agent i chooses to delete a connection the payoff is:

$$\pi_{i,t+1}(d) = u_i(g_t - ij)$$

[2]This choice rule is related to a distribution in classical statistical mechanics describing the velocities of particles in a gas.

The agents then update their payoff assessments using the following update rule:

$$Q^i_{t+1}(a_i) = \begin{cases} (1-\alpha)Q^i_t(a_i) + \alpha\pi_{i,t+1}(a_i) \text{ if } a_i \text{ is chosen.} \\ Q^i_t(a_i) \text{ otherwise .} \end{cases}$$

This reinforcement learning rule has been studied in [22] and [16], among others. The parameters include the initial assessments, Q^i_0, and the learning rate, $0 < \alpha < 1$. The learning rate determines how much previous payoffs are weighted in the current assessments. If α is close to one then payoffs in the distance past hold little weight in the current assessment . In fact, if α is close to one, the current assessment is very close to the last observed payoff.

Combining these transition probabilities, the each agent's Q-values change with the following probabilities:

$$Q^i_{t+1}(a) = \begin{cases} (1-\alpha)Q^i_t(a) + \alpha u_i(g_t + ij) \text{ with prob } p_{i,t}p_{j,t} \\ (1-\alpha)Q^i_t(a) + \alpha u_i(g_t - ij) \text{ with prob } p_{i,t}(1-p_{j,t}) \\ Q^i_t(a) \text{ with probability } 1 - p_{i,t} \end{cases}$$

$$Q^i_{t+1}(d) = \begin{cases} (1-\alpha)Q^i_t(d) + \alpha u_i(g_t - ij) \text{ with prob } 1 - p_{i,t} \\ Q^i_t(d) \text{ with prob } p_{i,t} \end{cases}$$

Given these transition probabilities, the next Lemma will give an upper and lower bound for the payoff assessments as time grows large.

Lemma 2. *Provided* $0 < \alpha < 1$*, the Q-values are asymptotically bounded between the minimum and maximum possible payoffs received for choosing action* a_i*, that is,*

Proof. First, notice the following observation which is obtained by induction.

$$Q_T^i(a_i) = (1-\alpha)^T Q_0^i(a_i) + \sum_{t=1}^{T} \alpha(1-\alpha)^{T-t} u_i(gt+1)$$

To see this, begin with the following:

$$Q_1^i(a_i) = (1-\alpha)Q_0^i(a_i) + \alpha r_1$$

Now assume that $Q_{T-1}^i(a_i) = (1-\alpha)^{T-1}Q_0^i(a_i) + \sum_{t=1}^{T-1} \alpha(1-\alpha)^{T-1-t} r_t$ holds. The following expression is obtained for $t = T$:

$$\begin{aligned} Q_T^i(a_i) &= (1-\alpha)Q_{t-1}^i(a_i) + \alpha r_T \\ &= (1-\alpha)(1-\alpha)^{T-1}Q_0^i(a_i) + \sum_{t=1}^{T-1} \alpha(1-\alpha)^{T-1-t} r_t + \alpha r_T \\ &= (1-\alpha)^T Q_0^i(a_i) + \sum_{t=1}^{T} \alpha(1-\alpha)^{T-t} r_t \end{aligned}$$

As $T \to \infty$, the first term approaches zero. The second term becomes a geometric series with ratio $(1-\alpha)$ where each term is multiplied by αr_i. For any values of N, δ and c in the connections model, there is a maximum and minimum value for u_i. Therefore, we must have for all t, $\min_g u_i(g) \leq r_t \leq \max_g u_i(g)$. Using this fact,

$$\sum_{t=1}^{T} \alpha(1-\alpha)^{T-t} \min_{g(a_i)} u_i(g(a_i)) \leq \sum_{t=1}^{T} \alpha(1-\alpha)^{T-t} r_t \leq \sum_{t=1}^{T} \alpha(1-\alpha)^{T-t} \max_{g(a_i)} u_i(g(a_i))$$

Where $g(a_i)$ denotes the set of possible networks formed after action a_i has been chosen by agent i. This distinction must be provided because in the connections, links cannot be formed unilaterally, the set of networks obtained when *delete* is chosen differ from those formed after *add* is chosen. Using the fact that $0 < \alpha < 1$ and the convergence of the geometric series, the following is true:

$$\min_{g(a_i)} u_i(g(a_i)) \leq \lim_{T\to\infty} Q_T^i(a_i) \leq \max_{g(a_i)} u_i(g(a_i))$$

Rewriting this in a more convenient form yields:

$$\min \pi_t(a_i) \leq \lim_{t\to\infty} Q_t^i(a_i) \leq \max \pi_t(a_i) \tag{3.3}$$

$\square$.

In reference to the symmetric connections model, we have the following asymptotic bounds for the Q-values. If $\delta > c$ and $\delta - c > \delta^2$, then

$$0 \leq \lim_{t\to\infty} Q_t^i(d) \leq (N-2)(\delta - c) + \delta^2$$
$$0 \leq \lim_{t\to\infty} Q_t^i(a) \leq (N-1)(\delta - c)$$

Whenever $\delta - c > \delta^2$, direct connections net cost are worth more than indirect connections. Therefore, the maximum payoff occurs when an agent maintains the maximum number of direct connections. For a network of size N, the maximum number of connections an individual can maintain is $N-1$ and therefore, the maximum payoff is $(N-1)(\delta - c)$. However, this payoff may only be obtained after choosing to add a connection, since connections may not be unilaterally added. The maximum payoff obtained from deleting a connection is $(N-2)(\delta - c) + \delta^2$. The minimum payoff occurs when an agent maintains has no connections and this payoff equal to zero.

If $\delta > c$, $\delta - c < \delta^2$, then the following bounds hold for the Q-values.

$$0 \leq \lim_{t\to\infty} Q_t^i(d) \leq (\delta - c) + (N-2)(\delta^2)$$

$$0 \leq \lim_{t\to\infty} Q_t^i(a) \leq (\delta - c) + (N-2)(\delta^2)$$

Whenever $\delta - c < \delta^2$, indirect connections may be worth more than direct connections net cost. Since $0 < \delta < 1$, the most valuable indirect connections give a payoff of δ^2 since $\delta^2 > \delta^k$ for any $k > 2$. However, to receive any benefits from an indirect connection, agents must maintain at least one direct connection. Therefore, the maximum possible payoff is $\delta - c + (N-2)\delta^2$. Note that this payoff may be received after adding or deleting a connection. Again, the minimum payoff occurs when an agent maintains no connections and is zero.

If $\delta < c$ and $c < \delta + (N-2)\delta^2$, then

$$(N-2)(\delta - c) \leq \lim_{t\to\infty} Q_t^i(d) \leq (\delta - c) + (N-2)(\delta^2)$$

$$(N-1)(\delta - c) \leq \lim_{t\to\infty} Q_t^i(a) \leq (\delta - c) + (N-2)(\delta^2)$$

Whenever $\delta < c$ and $c < \delta + (N-2)\delta^2$ direct connections always yield a negative payoff. The minimum payoff occurs when an individual maintains the maximum number of direct connections without any indirect connections. Since connections may not be added unilaterally, the minimum possible payoff received after deleting a connection is larger than the minimum possible payoff received after adding a connection. These two minimum payoffs are given by $(N-2)(\delta - c)$ and $(N-1)(\delta - c)$, respectively. Since $c < \delta + (N-2)\delta^2$, the maximum payoff is positive and occurs when individuals maintain one direct connection with $(N-2)$ indirect connections of length 2.

If $\delta < c$ and $c > \delta + (N-2)\delta^2$, then

$$(N-2)(\delta - c) \leq \lim_{t\to\infty} Q_t^i(d) \leq 0$$

$$(N-1)(\delta - c) \leq \lim_{t\to\infty} Q_t^i(a) \leq 0$$

Whenever $\delta < c$ and $c > \delta + (N-2)\delta^2$, the minimum possible payoffs are the same as the previous case. However, the maximum possible payoff is now zero since the cost of maintaining a direct connection is higher than any possible benefit that would be received from indirect connections.

In order to characterize which networks form over time, the following definition will be useful.

Definition 13. *A network is an absorbing network if there exist values $Q_t^i(a), Q_t^i(d)$ where $i \in N$ and τ_t, such that $g_s = g_t$ with probability 1 for all $s \geq t$.*

Let us examine the evolution of the overall network as follows, consider at time t the total number of connections in the network, which we will denote by $|g_t|$. The total number of edges can only increase if a connection $ij \notin g_t$ is added at time t. That is, for some $ij \notin g_t$, $g_{t+1} = g_t + ij$, which occurs with probability $p_{i,t}p_{j,t}$. Furthermore, the probability that any two agents i and j are matched is $\frac{2}{n(n-1)}$. The probability that $ij \in g_{t+1}$, given that $ij \in g_t$ is $\frac{2}{n(n-1)} \sum_{ij \in g_t} p_{i,t}p_{j,t}$. Similarly, the probability that $ij \notin g_{t+1}$, given $ij \notin g_t$ is $\frac{2}{n(n-1)} \sum_{ij \notin g_t} p_{i,t}p_{j,t}$. Thus, the total probability that $|g_{t+1}| = |g_t|$ is given by $\frac{2}{n(n-1)} \sum_{ij \in g_t} p_{i,t}p_{j,t} + \frac{2}{n(n-1)} \sum_{ij \notin g_t} p_{i,t}p_{j,t}$. The other cases may be derived similarly.

Given the current network, $|g_{t+1}|$ evolves according to the following:

$$|g_{t+1}| = \begin{cases} |g_t| + 1 & \text{with probability } \sum\limits_{ij \notin g_t} (\dfrac{2}{n(n-1)}) p_{i,t}p_{j,t} \\ |g_t| & \text{with probability } \sum\limits_{ij \in g_t} \frac{2}{n(n-1)} p_{i,t}p_{j,t} + \sum\limits_{ij \notin g_t} \frac{2}{n(n-1)} (1 - p_{i,t}p_{j,t}) \\ |g_t| - 1 & \text{with probability } \sum\limits_{ij \in g_t} \frac{2}{n(n-1)} (1 - p_{i,t}p_{j,t}) \end{cases}$$

These transition probabilities can help characterize which, if any, network may be an absorbing network.

Proposition 13. *The only possible absorbing networks for the RL model are the complete network, the empty network, or a network with both a completely connected component and an empty component. In addition, a necessary condition for an absorbing network is* $\tau_s^i = 0$ *for all* $i \in N$ *and all* $s \geq t$*.*

Proof. Using the transition probabilities given in the previous section, the absorbing networks may be described by characterizing the total number of connections. In this case, $g_{t+1} = g_t$ with probability 1 and thus $|g_{t+1}| = |g_t|$ with probability 1. Therefore, the probability that $|g_{t+}| = |g_t| + 1$ and the probability that $|g_{t+}| = |g_t| - 1$ must both equal zero. In terms of the current network, one must have:

$$\sum_{ij \notin g_t} (\frac{2}{n(n-1)}) p_{i,t} p_{j,t} = 0 \tag{3.4}$$

and

$$\sum_{ij \in g_t} \frac{2}{n(n-1)} (1 - p_{i,t} p_{j,t}) = 0 \tag{3.5}$$

There are three cases which satisfy both of these equations:

Case 1: If $|g_t| = \frac{n(n-1)}{2}$ then g_t is the complete network. Furthermore, $\sum\limits_{ij \in g_t} (1 - p_{i,t} p_{j,t}) = 0$ which implies $p_{i,t} = p_{j,t} = 1$ for all $i, j \in N$.

Case 2: If $|g_t| = 0$, then g_t is the empty network. Furthermore, $\sum\limits_{ij \notin g_t} p_{i,t} p_{j,t} = 0$ which implies for each $i, j \in N$ either $p_{i,t} = 0$ or $p_{j,t} = 0$ or both. In particular, we may have at most one agent k, with $p_{k,t} \neq 0$.

Case 3: If $\sum\limits_{ij \in g_t} (1 - p_{i,t} p_{j,t}) = 0$ and $\sum\limits_{ij \notin g_t} p_{i,t} p_{j,t} = 0$. Then if $ij \in g_t$ but $ik \notin g_t$ for some i, we must have $p_{i,t} = p_{j,t} = 1$ and $p_{k,t} = 0$. Therefore, there is no connection involving k in g_t, otherwise $\sum\limits_{ij \in g_t} (1 - p_{i,t} p_{j,t}) > 0$. Thus, k has no links. We can therefore partition the network into a connected component and an empty component. $\square$.

In each case we have $p_{i,t} = 1$ or 0, which implies $\tau_t^i = 0$. Given this fact, the behavior of the network formation process when $\tau^i = 0$ for each $i \in N$ should be examined. $\square$

Proposition 14. *Suppose $\delta > c$ in the connections model. If $\tau_t = 0$ and $Q_t^i(d) > Q_t^i(a)$, then $Q_s^i(d) \leq Q_s^i(a)$ for some $s \geq t$ unless $Q_t^i(a) < 0$.*

Proof. If $Q_t^i(a) \geq Q_t^i(d)$, then $s = t$. Otherwise, suppose at time t, τ_t^i is zero. If agent i has $k_i(g_t) = k$ connections and $Q_t^i(d) > Q_t^i(a)$ then with probability 1, for some $T \in \mathbb{N}$, $k_i(g_{t+T}) = 0$ (provided the $Q_s^i(d) > Q_s^i(a)$ for all $s \in [t, t+T]$, otherwise the result is immediate). Agent i will have zero connections at time $t+T$ and $u_i(g_{t+T}) = 0$, which implies that $\Delta Q_{t+T+1}^i = -\alpha Q_{t+T}^i$. Furthermore, $Q_s^i(d) = (1-\alpha)^s Q_{t+k}^i(d)$ for any $s > t+T$. As s grows large, $Q_s^i(d)$ approaches zero. Therefore, $Q_s^i(d) > Q_s^i(a) = Q_t^i(a)$ only if $Q_t^i(a) < 0$. In other words, if $\tau_t^i = 0$, agent i will only choose delete forever, if $Q_t^i(a) < 0$. This in turn implies agent i cannot have zero connections in the underlying network indefinitely. $\square$

Corollary 3. *If $\delta > c$ in the connections model, a network in which an agent has zero connections cannot be absorbing for the RL model.*

This is an immediate consequence of the above proposition and Lemma 1, which states $0 \leq \lim\limits_{t \to \infty} Q_t^i(a)$ for all $i \in N$. $\square$

Proposition 15. *Suppose $\delta < c$ in the connections model. If $\tau_t = 0$ and $Q_t^i(a) > Q_t^i(d)$ for each agent i at some time t, then $\exists\ s \geq t$ such that $Q_s^i(d) > Q_s^i(a)$ unless $Q_t^i(d) < (N-1)(\delta - c)$ for all $i \in N$.*

Proof. The payoff to agent i for maintaining $N-1$ connections ($k_i = N-1$) is $(N-1)(\delta - c)$. If agent i has $k_i(g_t)$ connections and then with probability 1, for some $T \in \mathbb{N}$, $k_i(g_{t+T}) = N-1$ for each agent i(provided the $Q_s^i(a) > Q_s^i(d)$ for all $s \in [t, t+T]$ and all $i \in N$ otherwise the result is immediate). Furthermore, given that $p_{i,t} = 1$, the change in $Q_t^i(d)$ is zero and since $u_i(g_t + ij) = (N-1)(\delta - c)$, we have $\Delta Q_t^i(a) = \alpha((N-1)(\delta - c) - Q_t^i(a))$. These relationships hold for all agents in the network. Therefore, if $p_{i,t} = p_{j,t} = 1$ for any $i, j \in N$, then s periods later we have $Q_{t+s}^i(a) = (1-\alpha)^s Q_t^i(a) + \sum_{l=1}^{s} \alpha(1-\alpha)^{s-l}(N-1)(\delta - c)$. Thus, as s grows large we have $Q_{t+s}^i(a) \approx (N-1)(\delta - c)$ for each agent i. Therefore, $Q_s^i(a) > Q_s^i(d) = Q_t^i(d)$ only if $Q_t^i(d) < (N-1)(\delta - c)$ for all agents $i \in N$. In other words, if $\tau_t^i = 0$ and $\delta < c$ each agent in the network will only choose add forever if $Q_t^i(d) < (N-1)(\delta - c)$ for all agents $i \in N$. This implies that the complete network cannot occur indefinitely unless the conditions in the proposition are met. □

Corollary 4. 1 *If $\delta - \delta^2 > c$ in the connections model, the complete network will form and remain in the RL model.*

This is an immediate consequence of Proposition 15 and Lemma 1, which states $0 \leq \lim_{t \to \infty} Q_t^i(d) \leq (N-2)(\delta - c)$ for all $i \in N$. □

Corollary 5. 1 *If $\delta < c$ in the symmetric connections model, the complete network is not a absorbing network for the RL model.*

Again, this is an immediate consequence of the above proposition and Lemma 1, which states whenever $\delta < c$ then $(N-2)(\delta - c) \leq \lim_{t \to \infty} Q_t^i(d) \leq 0$ for all $i \in N$. □

Proposition 16. *If $\delta > c$ in the connections model, the complete network may be absorbing for the RL model.*

Proof. By Proposition 14, if $Q_t^i(d) > Q_t^i(a)$, then for some time $s \geq t$, $Q_s^i(a) > Q_s^i(d)$ provided $Q_t^i(a) \geq 0$. By Proposition 15, the condition $Q_t^i(a) > Q_t^i(d)$ for all agents $i \in N$ will hold for all $s \geq t$ if $Q_t^i(d) < (N-1)(\delta - c)$. By Lemma 2, both of these bounds are met in the limit $t \rightarrow \infty$. Note that the all the absorbing networks other than the complete network have at least one agent with no connections. This implies the complete network is the only "absorbing" network in the limit whenever $\delta - \delta^2 > c$. □

Putting these propositions, we see that whenever $\delta > c$ the only possible absorbing network is the complete network. Whenever, $\delta < c$ we can only rule out the complete network. To examine whether the process converges to a absorbing network, a different approach must be taken. This question will be examined in the next section as well as through simulations in section 6.

E. Partial Observability

In the section, I will propose two informational settings. In the simple model considered above, agents had no information. However, agents may have access to some information. For instance, they may recognize the individuals they have been randomly matched with or be aware of the number of connections they maintain. I will use a state-contingent reinforcement learning algorithm (similar to Q-learning) to capture both these ideas. The model is the as same as the one presented in the previous section with one distinction: agents observe the individual they have been randomly matched with or the number of connections they currently have. In both situations, each agent maintains two Q-values (one for each action) for each agent they may be matched with (or number of connections they have) for a total of $2(n-1)$ Q-values. The learning rule is now given by the following:

$$Q^i_{t+1}(j,a) = \begin{cases} (1-\alpha)Q^i_t(j,a) + \alpha(u_i(g_{t+1})) \text{ if action a is chosen} \\ \\ Q^i_t(j,a) \text{ otherwise} \end{cases}$$

where $j \in \{1, \ldots, n\}$ denotes the agent matched with agent i at time t and α is the learning rate.

Furthermore, the probability with which the agent chooses his action is now state dependent and we will denote the probability that agent i chooses to add a connection with agent j by $p_{i,t}(j)$, which is defined as follows:

$$p_{i,t}(j) = \frac{e^{Q^i_t(j,a)/\tau^i_t}}{e^{Q^i_t(j,a)/\tau^i_t} + e^{Q^i_t(j,d)/\tau^i_t}} \tag{3.6}$$

where τ^i_t is same the rationality parameter as in the previous model. As before, we will assume that $\tau^i_t \to 0$ as $t \to \infty$.

Lemma 3. *Provided* $0 < \alpha < 1$*, the Q-values are asymptotically bounded between the minimum and maximum possible payoffs received for choosing action* a_i*, that is,*

$$\min \pi_t(j, a_i) \leq \lim_{t\to\infty} Q^i_t(j, a_i) \leq \max \pi_t(j, a_i) \tag{3.7}$$

Proof. Using the same argument as Lemma 2, it can shown that limiting values of the payoff assesments for each state must fall between the minimum and maximum possible payoffs occuring in that state. □

As before, the transition probabilities related to the underlying network can be used to determine the set of possible absorbing networks.

$$|g_{t+1}| = \begin{cases} |g_t|+1 & \text{with probability } \sum_{ij\notin g_t} \frac{2}{n(n-1)})p_{i,t}(j)p_{j,t}(i) \\ |g_t| & \text{with probability } \sum_{ij\in g_t} \frac{2}{n(n-1)}p_{i,t}(j)p_{j,t}(i) + \sum_{ij\notin g_t} (\frac{2}{n(n-1)})(1-p_{i,t}(j)p_{j,t}(i)) \\ |g_t|-1 & \text{with probability } \sum_{ij\in g_t} \frac{2}{n(n-1)}(1-p_{i,t}(j)p_{j,t}(i)) \end{cases}$$

Proposition 17. *Any network may be an absorbing network for the RLA model.*

Proof. Let g_t be a network at time t. Then $g_{t+1} = g_t$ with probability 1 if and only if $|g_{t+1}| = |g_t|$ with probability 1. This implies that $\sum_{ij\notin g_t} \frac{2}{n(n-1)}p_{i,t}(j)p_{j,t}(i) = 0$ and $\sum_{ij\in g_t} \frac{2}{n(n-1)}(1-p_{i,t}(j)p_{j,t}(i)) = 0$. In this case, since the probabilities are state dependent, any network as a absorbing network by required $p_{i,t}(j) = p_{j,t}(i) = 1$ for $ij \in g_t$ and either $p_{i,t}(j) = 0$ or $p_{j,t}(i) = 0$ or both for $ij \notin g_t$. □

Proposition 18. *If $\tau_t^i = 0$, $\delta > c$, and $Q_t^i(j,d) > Q_t^i(j,a)$ for some i and all $j \in N$, then $Q_s^i(j,d) \leq Q_s^i(j,a)$ for some $s \geq t$ unless $Q_t^i(j,a) < 0$.*

Proof. If $Q_t^i(j,d) > Q_t^i(j,a)$ for some i and all $j \in N$ and $\tau_t^i = 0$, then $p_{i,t}(j) = 0$. If agent i has one or more connections at time t, they will sever all their connections with probability 1 and continue to maintain zero connections. Their payoff assessments for each state, $Q_s^i(j,d)$ will decrease towards zero and, unless $Q_t^i(j,a) < 0$, $Q_s^i(j,d) \leq Q_s^i(j,a)$ for some time s. □

Corollary 6. *If $\delta > c$ in the connections model, a network in which an agent has zero connections is not an absorbing network.*

This corollary is an immediate consequence of Lemma 3 and Proposition 18.□

Proposition 19. *If $\tau_t^i = 0$ and $Q_t^i(j,a) > Q_t^i(j,d)$ for each agent $i, j \in N$, then $Q_s^i(j,d) > Q_s^i(j,a)$ for some $s \geq t$ unless $Q_t^i(j,d) < (N-1)(\delta - c)$ for all $i, j \in N$.*

Proof. If $Q_t^i(j,a) > Q_t^i(j,d)$ for each pair of agents $i,j \in N$ and $\tau_t^i = 0$ then $p_{i,t}(j) = p_{j,t}(i) = 1$. Therefore, at some time $s \geq t$ each agent will have $(N-1)$ connections. Using the same argument as before, each payoff assesment, $Q_s^i(j,a)$, will approach $(N-1)(\delta - c)$ as s increases. Therefore, unless $Q_t^i(j,d) < (N-1)(\delta - c)$ for each pair of agents i,j, $Q_s^I(j,a) > Q_s^i(j,a)$ for all $s \geq t$. □

Corollary 7. *If $\delta < c$ in the symmetric connections model, the complete network is not an absorbing network.*

This corollary is an immediate consequence of Proposition 19 and Lemma 3. □

Corollary 8. *If $\delta - \delta^2 > c$ in the symmetric connections model, the complete network is an absorbing network.*

This corollary is also an immediate consequence of Proposition 19 and Lemma 3. □

Proposition 20. *The set of absorbing networks for the RLC model is the same as the RL model.*

Proof. Let g_t be a network at time t. Then $g_{t+1} = g_t$ with probability 1 if and only if $|g_{t+1}| = |g_t|$ with probability 1. This implies that number of connections each agent maintains is constant. Suppose agent i currently has deg_i connections, if $p_i(deg_i, t) = 0$, then eventually deg_i will decrease provided $deg_i \neq 0$. Therefore, for any absorbing network $p_i(deg_i, t) = 1$ or $deg_i = 0$.

If $P_j(deg_i, t) = 1$ and there exists an agent j with $p_j(deg_j, t) = 1$ such that $ij \notin g_t$, eventually these agents will meet and both agents will choose to add the connection. Therefore, for any absorbing network each pair of agents i,j such that $p_i(deg_i, t) = 1$ and $p_j(deg_j, t) = 1$, $ij \in g_t$. Together with the previous statement, this implies that a absorbing network for the RLC must consist of a completely connected component and an empty component. This set of networks coincides with the set of absorbing networks for the RL model. □

Proposition 21. *If $\tau_t^i = 0$, $\delta > c$, $k_i = 0$, and $Q_t^i(0, d) > Q_t^i(0, a)$ for some i, then $Q_s^i(0, d) \leq Q_s^i(0, a)$ for some $s \geq t$ unless $Q_t^i(0, a) < 0$.*

Proof. If $\tau_t^i = 0$, $\delta > c$, $k_i = 0$, and $Q_t^i(0, d) > Q_t^i(0, a)$ for some i, then agent i will continue to maintain zero connections in the network and the assessment $Q_s^i(0, d)$ will approach zero. If $Q_t(0, a) > 0$, then for some time s, $Q_s^i(0, d) \leq Q_s^i(0, a)$. □

Corollary 9. *If $\delta > c$ in the connections model, a network in which an agent has zero connections is not an absorbing network.*

This corollary is an immediate consequence of Lemma 3 and Proposition 21.□

Proposition 22. *If $\tau_t^i = 0$, $Q_t^i(N-1, a) > Q_t^i(N-1, d)$, and $deg_i = N-1$ for each agent i, then $Q_s^i(N-1, d) > Q_s^i(N-1, a)$ for some $s \geq t$ unless $Q_t^i(N-1, d) < (N-1)(\delta - c)$ for all $i \in N$.*

Proof. If $\tau_t^i = 0$, $Q_t^i(N-1, a) > Q_t^i(N-1, d)$, and $deg_i = N-1$ for each agent i, then $p_i(N-1, a) = 1$ for all agents $i \in N$. The payoff assesment $Q_s^i(N-1, a)$ will approach $(N-1)(\delta - c)$ as s grows. Therefore, unless $Q_t^i(N-1, d) < (N-1)(\delta - c)$ for all $i \in N$, eventually $Q_s^i(N-1, d) > Q_s^i(N-1, a)$. □

Corollary 10. *If $\delta < c$ in the symmetric connections model, the complete network is not an absorbing network.*

This corollary is an immediate consequence of Lemma 3 and Proposition 22. □

Corollary 11. *If $\delta - \delta^2 > c$ in the symmetric connections model, the complete network is an absorbing network.*

This corollary is also an immediate consequence of Lemma 3 and Proposition 22.
□

Traditionally, the long-run behavior of reinforcement learning has been analyzed using stochastic approximation, by approximating the learning dynamics with a continuous time limit, or using properties of Markov chains. See, for example, [2], [18], and [3]. Unfortunately, these approaches encounter difficulty whenever the payoffs are state-dependent, as is the case for each of the models discussed so far. In order to analyze the behavior of these models, I will turn to simulations.

F. Simulations

In this section, I examine the long-run behavior of the RL, RLA, and RLC models. The number of possible networks increases exponentially with the number of agents and limits my simulations to a relatively low number of agents. I examine the complete distribution over networks for simulations with $n = 4$ and $n = 5$ agents. In order to simulate each model the following need to be specified: the initial Q-values for each agent, the parameters of the connections model, δ and c, the learning rate α, choice parameter τ_t, and the initial network g_0. Given this information and the payoff function defined in the connections model, the model may be simulated. 1000 simulations were performed for each case, so that no initial Q-values were overweighted. The rationality parameter τ_t used in the simulations gradually decreased over time according to the following rule:

$$\tau_t = .9999^t$$

This ensures the rationality parameter is close to one at the beginning of the simulations and very close to zero as t grows large. A similar rule has been used in other studies on Q-learning, including, for example, [27] and [21].

G. Results of the Simulations

1. Effect of the parameters δ and c

Together the parameters δ and c determine which networks are pairwise stable and efficient. As discussed in Section C, there are three regions with different networks exhibiting stability and efficiency. The regions of efficiency and the associated network are shown in Fig. 9 and the different regions of pairwise stability and the associated networks are given in Fig. 10.

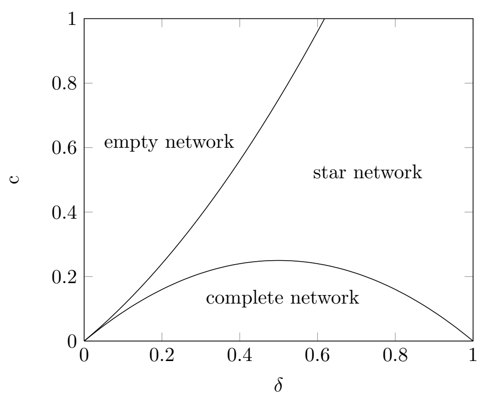

Fig. 9. Regions of efficiency

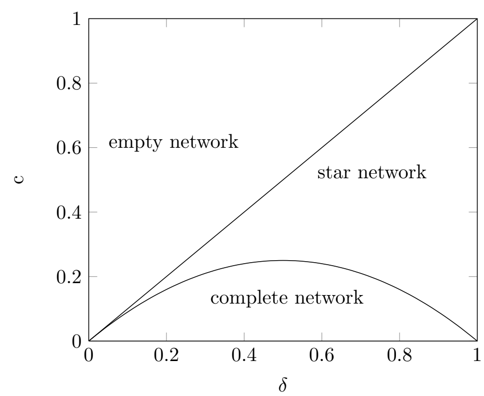

Fig. 10. Regions of pairwise stability

The simulations used the following parameter ranges $\delta \in [.1, .2, \ldots, .9]$ and $c \in [.1, .2, \ldots, .9]$. The initial Q-values were uniformly distributed on the interval $[u_{min}, u_{max}]$ where u_{min} and u_{max} represent the minimum and maximum payoff an agent receives under the symmetric connections model, respectively. A second framework was considered in which the agents were initially unbiased with all the initial

Q-values equal to zero, but the results of the simulations were not affected. Two frameworks for the starting network were also considered. As in [29], in the first scenario, the initial network was the empty network. In the second case, the initial network was chosen randomly with all possible networks being equally likely. Again, the results of the simulations were not affected. In what follows, I report the results of the simulations in which the initial Q-values and the starting network were random as described above. The simulations were repeated 1000 times and the relative frequency for each network was recorded for the each model. The relative frequency or empirical likelihood of a network, g^*, at time t is defined the number of times $g_t = g^*$ divided by the number of simulation runs.

a. RL Model

Fig. 11 shows the relative frequency of the empty network at $t = 10,000$ for the various combinations of δ and c, while Fig. 12 gives the relative frequency for the complete network.

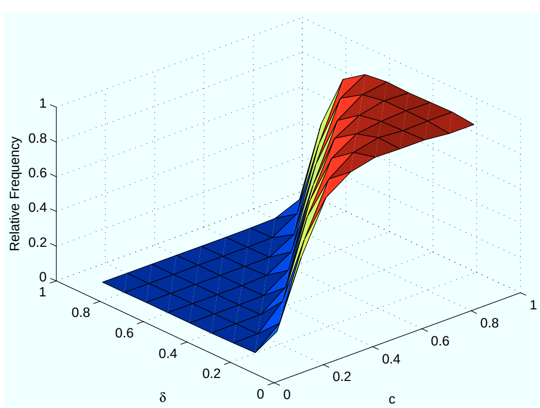

Fig. 11. RL: Rel. freq. of the empty network

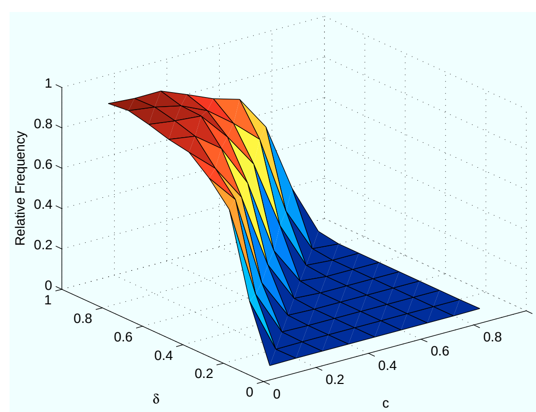

Fig. 12. RL: Rel. freq. of the complete network

The simulations for the RL model imply that the likelihood of the network being complete network is highest whenever $\delta > c$ and the probability of empty network when $c > \delta$. The relative frequency of the these networks increases as the difference between δ and c increases. The result of the simulations support the analytical results for the RL model. The relative frequencies of all other networks were very small compared to the empty and the complete networks.

b. RLA Model

The following figures show the relative frequency for the various combinations of δ and c for the RLA model. When the relative frequencies of the various networks are compared to the regions illustrated in Figures 9 and 10, a pattern emerges. The empirical likelihood for the empty network, shown in Fig. 13, is very close to one whenever $c > \delta + \delta^2$. In this case, the empty network is not only pairwise stable, but also efficient. The empirical probability of complete network, shown in Fig. 14, is largest whenever $c < \delta - \delta^2$. In this region, the complete network is both pairwise stable and efficient. One interesting observation regarding the simulation of the RLA model is the fact that the empirical likelihood of the empty network is very close to one whenever $c > \delta + \delta^2$, however, the empirical likelihood of the complete network is relatively low in the region in which it is both pairwise stable and efficient. The empirical likelihood for the star network (shown in Fig. 15, Fig. 16, Fig. 17, and Fig. 18) is low, in general, but a star network is most likely to form when $\delta - \delta^2 < c < \delta + \delta^2$.

c. RLC Model

The following figures show the relative frequency for the various combinations of δ and c for the RLC model. Patterns similar to those the simulations of the RLA model are seen for the RLC model. The empirical probability of the empty network

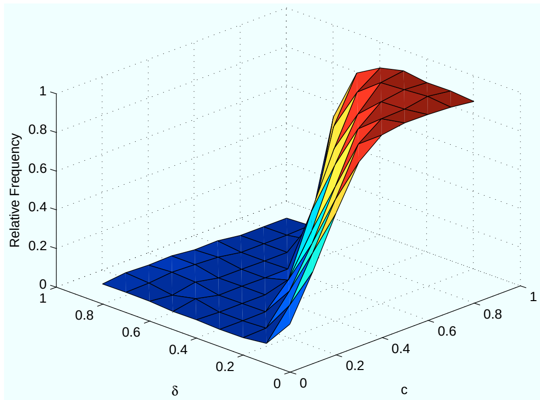

Fig. 13. RLA: Rel. freq. of the empty network

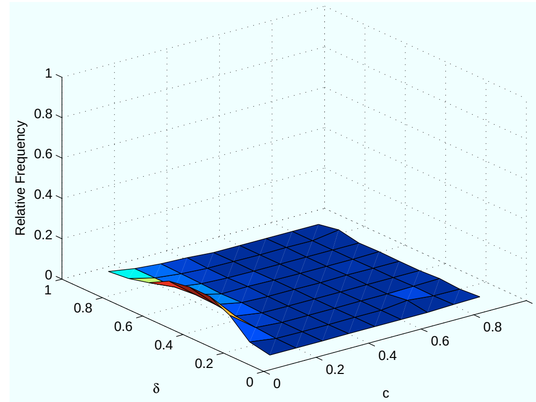

Fig. 14. RLA: Rel. freq. of the complete network

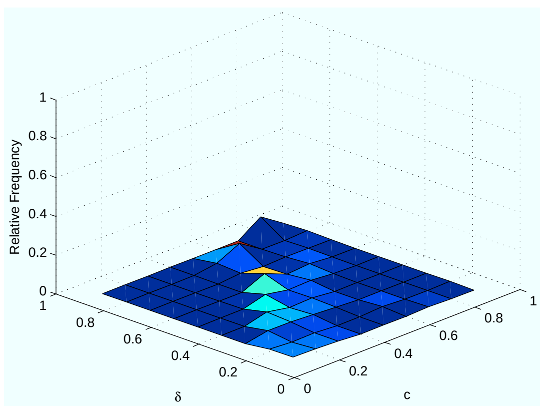

Fig. 15. RLA: Rel. freq. of star network 1

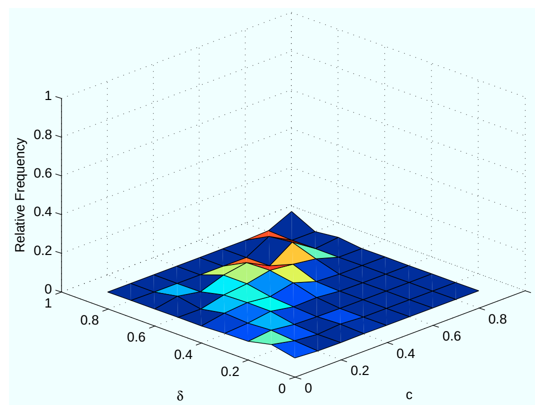

Fig. 16. RLA: Rel. freq. of star network 2

(Fig. 19) is largest whenever $c > \delta + \delta^2$. The empirical probability of the complete network (Fig. 20) highest whenever $c < \delta - \delta^2$. In contrast to the RLA model, the empirical probability of the star network (Fig. 21 and Fig. 22) is very low when $\delta - \delta^2 < c < \delta + \delta^2$.

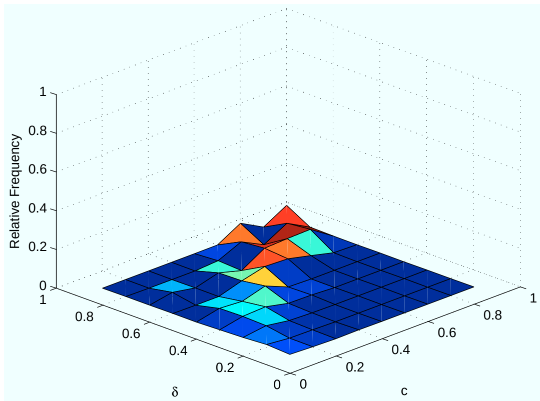

Fig. 17. RLA: Rel. freq. of star network 3

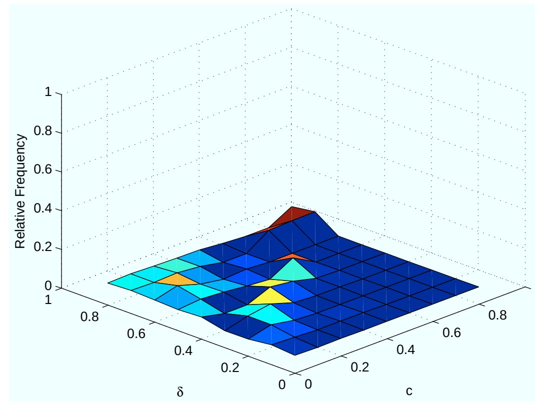

Fig. 18. RLA: Rel. freq. of star network 4

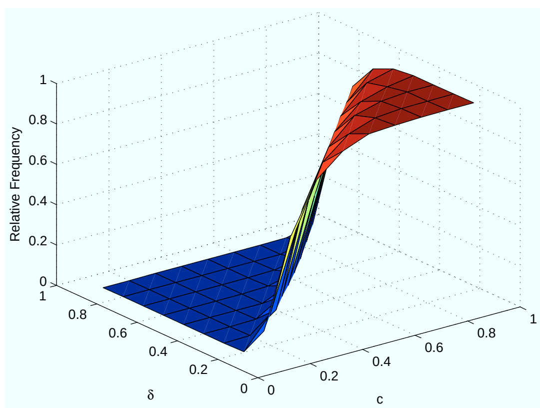

Fig. 19. RLC: Rel. freq. of the empty network

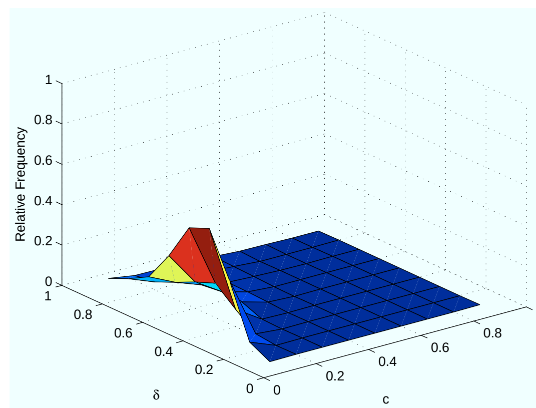

Fig. 20. RLC: Rel. freq. of the complete network

2. Effect of the learning rate α

The learning rate α determines how much past payoffs are weighted. High alpha imply the last observed payoff carry much more weight than the distant past. My simulations show that low values of α may slow the rate of convergence of the network formation process. The following figures show the effect of varying alpha on the relative frequency of certain networks for the given values of δ and c.

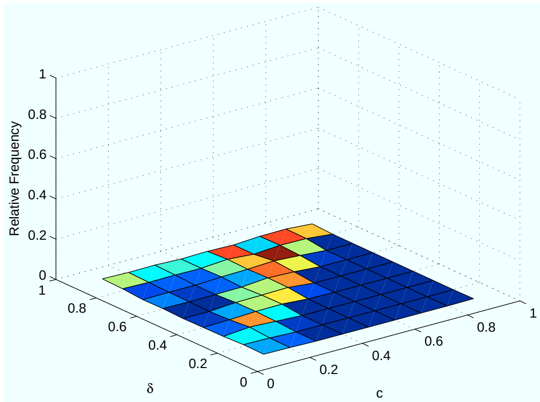

Fig. 21. RLC: Rel. freq. of star network 1

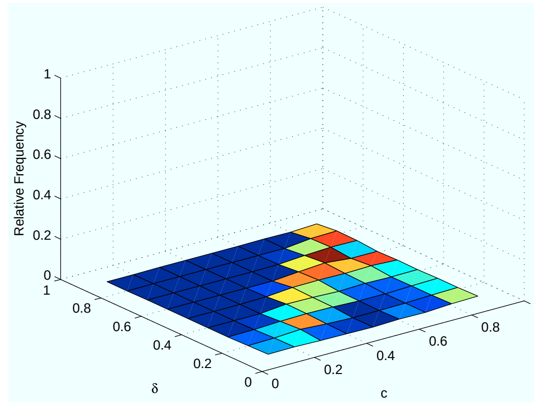

Fig. 22. RLC: Rel. freq. of star network 2

a. RL Model

The simulations show that as the learning rate alpha increases, the probability of forming a network other than the complete or empty network decreases significantly. In Fig. 23 the relationship between the relative frequency of the empty network and the learning rate is examined in the region in which $\delta \leq c$. In general, there seems to be a positive relationship between the two. The relationship between the relative frequency of the complete network and the learning rate in the region in which $\delta - \delta^2 > c$ is shown in Fig. 24. Again, there is a positive relationship between the two variables. Fig. 25 shows that the learning rate has a significant effect on the relative frequency of the complete network whenever $\delta - \delta^2 < c$. For low learning rates, the relative frequency of the complete network is also low. However, as the learning rate increase the relative frequency of the complete network increases in a nearly linear relationship. Intuitively, this makes sense because whenever the learning rate is low, agents give a larger weight to past payoffs. If an agent deletes a direct link with an agent, but is still indirectly connected to the same agent, their payoff will increase. Agents with “long memories” will remember this and adjust their probabilities more slowly than agents with "short" memories. If an agent with a short memory performs the same action and increases their probability of deleting links, they will delete more links and remember the payoffs associated with these actions which may be lower than the payoffs they received in the complete network. Hence, agents with a high learning rate may sever their links and observe the drop in payoff and quickly return to adding links with a high probability. Therefore, one might expect the relative frequency of the complete network to increase as alpha increases.

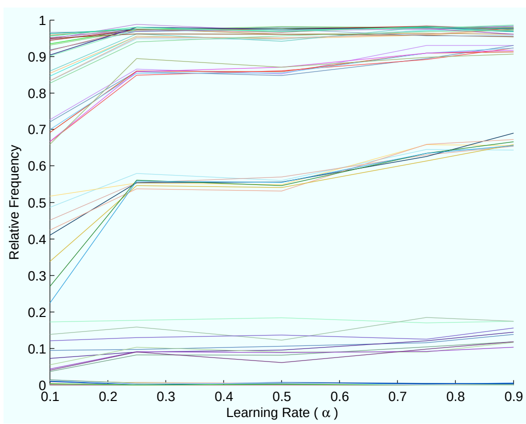

Fig. 23. RL Model: Rel. freq. of g^e vs α

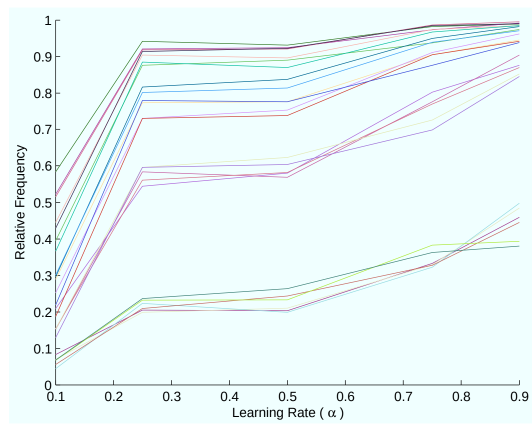

Fig. 24. RL Model: Rel. freq. of g^N vs α

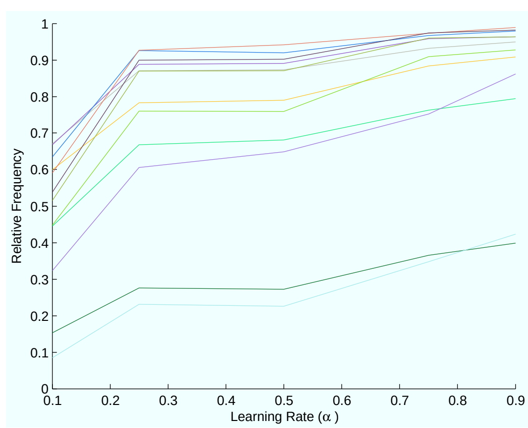

Fig. 25. RL Model: Rel. freq. of g^N vs α

b. RLA Model

For the simulations of the RLA model, the empty network is observed to have the highest empirical likelihood whenever $\delta < c$. In this region, the effect of the learning rate on the empirical likelihood of the empty network is illustrated in Fig. 26. In general, low values of α correspond lower empirical likelihood for the empty network than ceterus parebus high values of α. There is no clear relationship between the learning rate and the empirical likelihood of the complete network in the region in which $\delta - \delta^2 \leq c$ (Fig. 27) or $\delta - \delta^2 > c$ (Fig. 28).

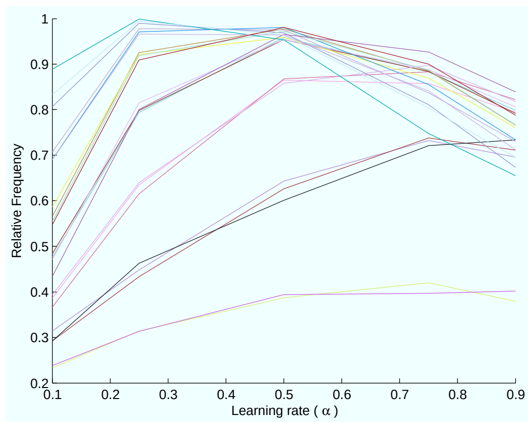

Fig. 26. RLA Model: Rel. freq. of g^e vs α

c. RLC Model

As with the RLA model, in the region in which the empty network is pairwise stable there is a positive relationship between the learning rate and the empirical likelihood of the empty network. Fig. 29 shows this relationship for the various values of δ and c which fall in this region. In the region $\delta - \delta^2 < c$, there is a very noisy relationship, as seen in Fig. 30, between the learning rate and the empirical likelihood of the wheel network, which had the largest empirical likelihood in many of the simulations within

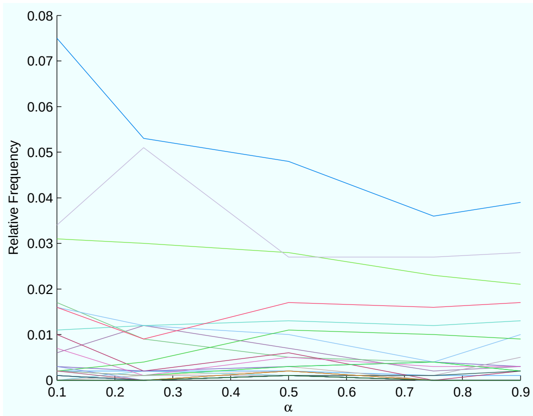

Fig. 27. RLA Model: Rel. freq. of g^N vs α

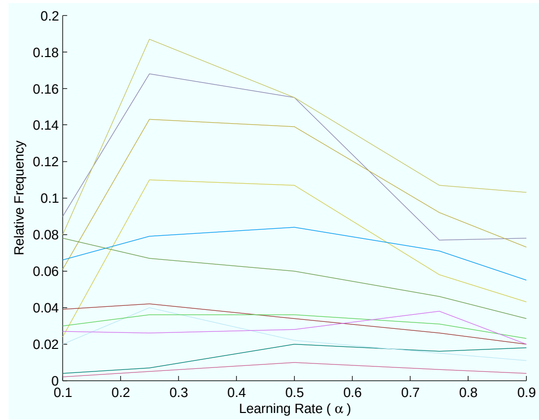

Fig. 28. RLA Model: Rel. freq. of g^N vs α

that region. In the region in which $\delta - \delta^2 > c$, the learning rate does not exhibit a consistent relationship effect with the empirical likelihood of the complete network, as shown in Fig. 31.

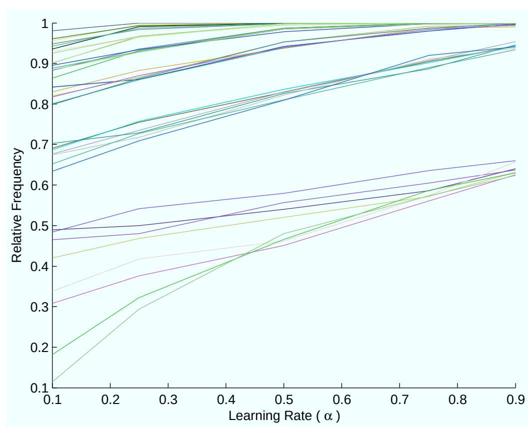

Fig. 29. RLC Model: Rel. freq. of g^e vs α

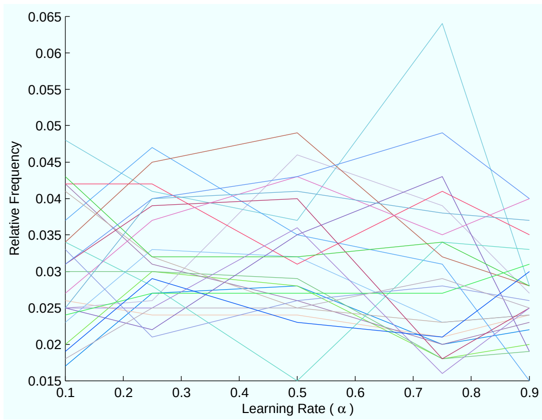

Fig. 30. RLC Model: Rel. freq. of g^W vs α

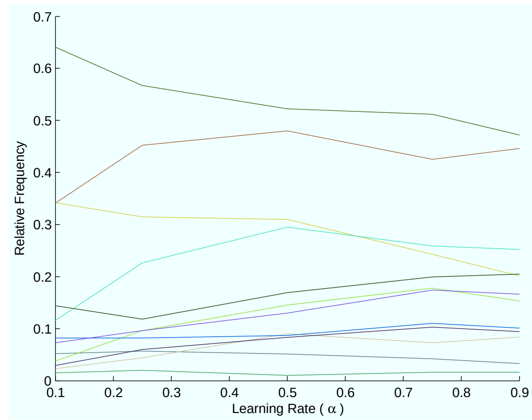

Fig. 31. RLC Model: Rel. freq. of g^N vs α

3. Networks most likely to form

As seen above, the empirical likelihood of the efficient network for the various simulations was quite low for some regions in the parameter space over δ and c. The network with largest empirical likelihood for each simulation is reported below.

For the RL model, the empty network has the largest empirical likelihood whenever $\delta \leq c$. The complete network has the largest empirical likelihood whenever $\delta > c$ and the empty network has the largest empirical likelihood when $\delta \leq c$. In Fig. 32, $\emptyset$ represents the empty network and g^N denotes the complete network. These results are supported by the analytical results proven in section 4.

The networks which are most likely to form for the RLA model illustrated in Fig. 33 where g^{star} denotes the star network. In the blank points, there was no clear pattern as to which network was the most likely to form and the networks were neither pairwise stable nor efficient. The empty network was the network most likely to form whenever $\delta < c$ and in some cases when $\delta = c$. The complete network was the mostly likely to form when $\delta - \delta^2 > c$, however, this relationship did not always

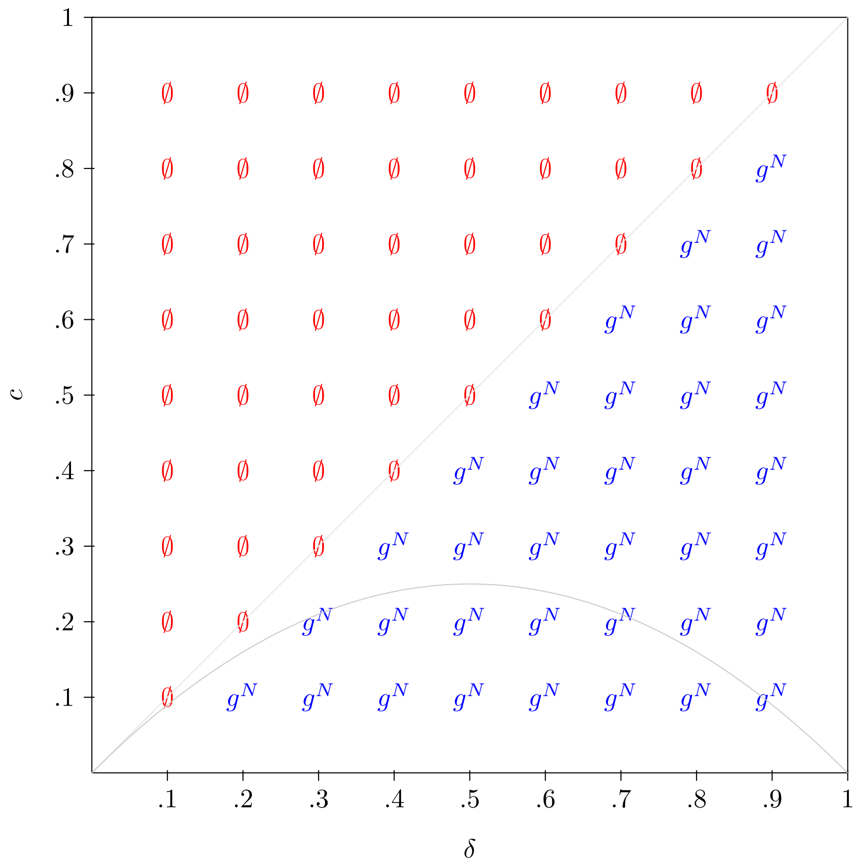

Fig. 32. RL Model: Networks most likely to form

hold. In the region in which $\delta - \delta^2 < c$ no clear pattern was observed.

Fig. 34 shows the networks that were most likely to form during the simulations of the RLC model. As with the RLA model, the empty network had the highest observed empirical likelihood when $\delta < c$ and for some cases when $\delta = c$. The complete network had the largest empirical likelihood whenever $\delta - \delta^2 > c$. Furthermore, when compared to the RLA model, this pattern was stronger in the sense that for only one simulation ($\delta = .3$, c=.2) did the complete network not have the highest observed frequency. An interesting pattern emerged in the region for which $\delta - \delta^2 < c < \delta$. Two

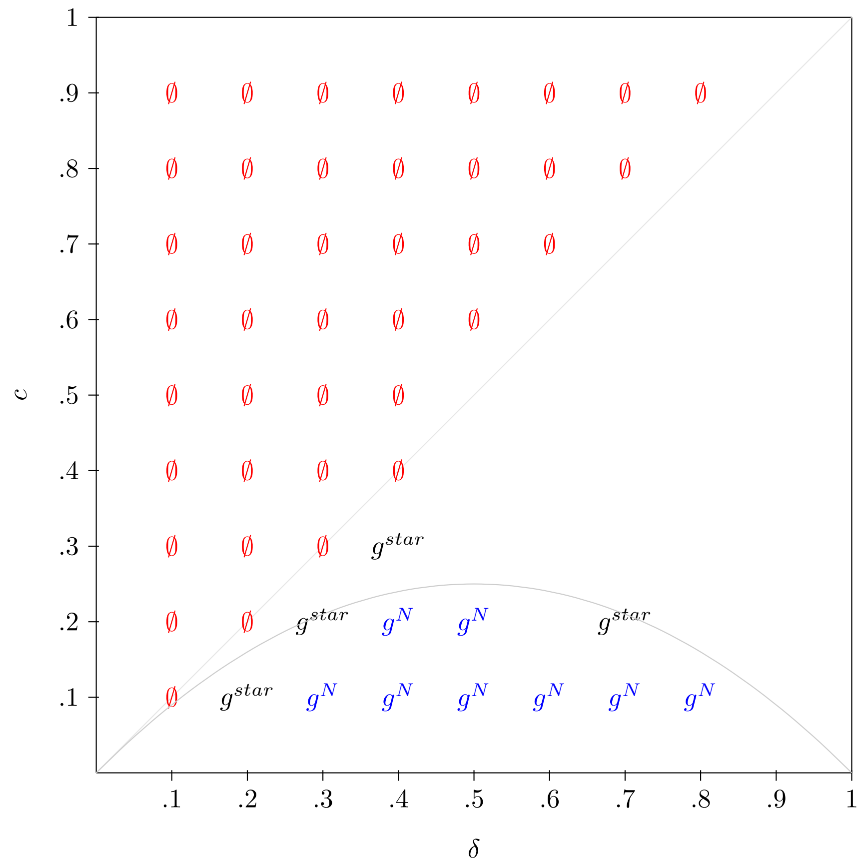

Fig. 33. RLA Model: Networks most likely to form

network topologies dominated this region, namely the wheel[3] and the chain network[4], denoted by g^W and g^c, respectively. As with the RLA model, there were regions in which no clear pattern emerged with regards to the network with the highest observed frequency.

[3]The wheel network is a network which n links in which in agents may be labeled such that $g^W = \{12, 23, \ldots, n1\}$.

[4]A chain network is a network with $n-1$ links in which agents may relabeled such that $g^c = \{12, 23, \ldots, (n-1)n\}$.

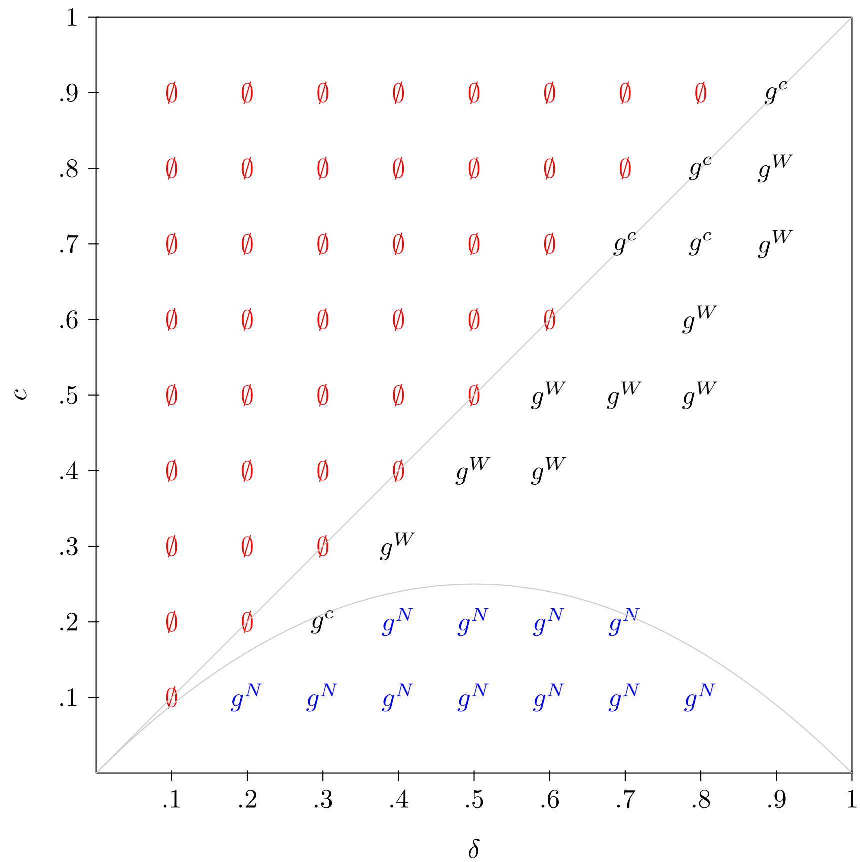

Fig. 34. RLC Model: Networks most likely to form

4. χ^2 two sample tests

Although it appears that each model produces different long-run empirical distributions with regards to networks, this needs to be quantified in some way. I will use the two-sample χ^2 test (see [7], Chapter 2) to test the hypothesis that the long-run behavior of the each model is the same. Under the two-sample χ^2 test, the null hypothesis is that the two mutually independent random samples are drawn from the same distribution with the alternative that the populations differ in some way. The χ^2-statistic for two samples of identical size is given by:

Table 1. χ^2 two-sample test: RLA vs RLC

	$\delta = .1$	$\delta = .2$	$\delta = .3$	$\delta = .4$	$\delta = .5$	$\delta = .6$	$\delta = .7$	$\delta = .8$	$\delta = .9$
$c = .1$	**66.21**	141.16	357.57	481.57	710.76	582.12	601.17	560.82	538.48
$c = .2$	169.63	**62.81**	**66.87**	334.54	842.36	914.86	666.67	707.09	621.75
$c = .3$	434.57	865.82	1468.4	1976.3	1978.3	1984.7	1724.2	1380.2	728.10
$c = .4$	119.98	304.50	193.60	93.58	451.02	707.70	780.58	1092.1	967.51
$c = .5$	**44.50**	139.48	308.23	337.03	523.11	669.18	873.94	1213.0	1133.7
$c = .6$	**22.83**	**50.83**	97.15	198.49	179.50	940.94	855.89	981.04	1223.8
$c = .7$	**26.29**	**45.70**	**53.16**	156.67	273.63	608.34	1312.4	1110.8	1016.6
$c = .8$	**31.49**	**21.15**	**30.34**	**50.23**	120.20	197.27	1660.0	1358.3	1084.0
$c = .9$	**48.13**	**37.62**	**38.59**	**15.81**	**47.96**	134.10	189.62	1874.1	1312.9

Notes: The bold face statistic represent a Q-statistic below the critical value and hence a failure to reject H_0.

$$Q = \sum_{i=1}^{r} \frac{(f_{i1} - f_{i2})^2}{f_{i1} + f_{i2}}$$

where f_{ij} denotes the number of observations classified as category i in the sample j and r is the total number of categories. It is known that this test-statistic has a χ^2 distribution with $r-1$ degrees of freedom under the null hypothesis. In what follows, $n = 4$, which means there 64 different networks and hence 63 degrees of freedom. The critical value for the the Q-statistic at the 1% confidence level is $Q_c = 92.01$. Table 1 tests the empirical distributions of the RLA model and the RLC model. Table 2 evaluates the relationship between the RL and RLC models and Table 3 compares the RL and RLA models.

The results of the χ^2 two sample tests seem to suggest that the long-run behavior of the RL and RLC is the same whenever $c > \delta$. The long-run behavior of each model is different whenever $\delta > c$. In the region in which $\delta - \delta^2 > c$, the empirical likelihood of the complete model is large for each model. This suggests the rate of convergence may differ for each model.

Table 2. χ^2 two-sample test: RL vs RLC

	$\delta = .1$	$\delta = .2$	$\delta = .3$	$\delta = .4$	$\delta = .5$	$\delta = .6$	$\delta = .7$	$\delta = .8$	$\delta = .9$
c=.1	**61.87**	335.15	333.39	209.07	384.70	756.06	1226.3	1469.6	1582.1
c=.2	**47.02**	**77.84**	499.22	760.86	760.69	1118.6	1353.4	1594.8	1644.4
c=.3	**28.05**	**43.48**	**84.83**	515.68	841.56	1155.8	1504.4	1541.4	1642.0
c=.4	**19.71**	**25.26**	**29.55**	123.47	506.89	957.56	1244.6	1423.6	1499.0
c=.5	**15.44**	**17.81**	**33.26**	**55.01**	171.91	643.85	1023.9	1230.6	1380.6
c=.6	**17.12**	**16.78**	**17.22**	**28.88**	**58.81**	229.66	672.10	1164.8	1335.7
c=.7	**21.71**	**21.98**	**25.41**	**13.05**	**29.30**	**71.86**	234.93	741.26	1164.5
c=.8	**39.78**	**20.34**	**21.27**	**13.91**	**18.04**	**38.16**	**88.61**	274.44	765.98
c=.9	**39.78**	**23.29**	**27.14**	**30.86**	**9.92**	**11.89**	**50.13**	**68.15**	252.79

Notes: The bold face statistic represent a Q-statistic below the critical value and hence a failure to reject H_0.

Table 3. χ^2 two-sample test: RL vs RLA

	$\delta = .1$	$\delta = .2$	$\delta = .3$	$\delta = .4$	$\delta = .5$	$\delta = .6$	$\delta = .7$	$\delta = .8$	$\delta = .9$
c=.1	**84.55**	482.71	820.94	883.38	1191.5	1309.4	1522.7	1642.6	1666.8
c=.2	151.66	**78.54**	545.82	1160.2	1475.2	1623.0	1653.5	1729.6	1734.3
c=.3	201.91	129.20	120.56	684.95	1221.8	1576.8	1774.2	1753.7	1768.0
c=.4	129.20	272.74	192.64	92.92	883.52	1321.3	1574.2	1771.9	1728.8
c=.5	**49.65**	139.19	322.48	330.43	564.98	1251.4	1576.5	1740.3	1784.6
c=.6	**37.11**	**64.41**	116.02	219.60	209.49	1074.0	1482.1	1638.7	1787.2
c=.7	**39.40**	**70.08**	**63.60**	165.56	302.91	642.58	1302.7	1387.5	1653.4
c=.8	**70.03**	**44.02**	**53.01**	**51.32**	140.34	207.46	1692.4	1421.9	1479.5
c=.9	**86.03**	**55.73**	**64.03**	**28.23**	**52.49**	129.84	259.67	1873.4	1359.0

Notes: The bold face statistic represent a Q-statistic below the critical value and hence a failure to reject H_0.

5. Summary of the simulations

The simulations discussed above suggest the following characteristics of the long-run behavior of the RL, RLA, and RLC models.

In the RL model, the network formation process converges to either the empty network or the complete network depending on whether $\delta > c$ or $\delta \leq c$. The rate of convergence is related to the magnitude of $|\delta - c|$. The learning rate α has a strong effect on the probability the complete network forms whenever $\delta > c$ and in particular when $\delta - \delta^2 < c$.

In the RLA model, the empty network forms with high probability whenever $\delta + \delta^2 < c$. The probability of the complete network forming is largest whenever $\delta - \delta^2 > c$ and the probability of the star network is largest whenever $\delta - \delta^2 < c < \delta + \delta^2$. However, the probability of the star network forming is very low in this region, due to the coordination problem of each agent choosing the same central agent.

In the RLC model, the empty network forms with high probability whenever $\delta < c$. As with the RLA model, the probability of the complete network forming is largest in the region $\delta - \delta^2 > c$. In the region, $\delta - \delta^2 < c < \delta + \delta^2$ the wheel network was observed to have the highest probability of forming. Within this region, the wheel network alleviates the coordination problem since it has a larger degree of symmetry than the star network.

The long run behavior of each model differs in the region $\delta > c$. The two-sample χ^2 test cannot distinguish between the behavior of the RL and RLC model whenever $\delta < c$. Furthermore, the behavior of the RLA is not found to be different than the RL or RLC model in regions in which c is much larger than δ. As noted in [29] and [13], in the region $\delta < c$, agents may experience strong economies of scale, in that the cost of the first link exceeds its benefits to the agents involved. However, the benefits of further links may exceed the cost.

H. Conclusion

In this paper, I develop a model of network formation in which agents decide whether or not to form links with others based on past experience and with varying degrees of additional information. I have shown that reinforcement learning agents need little information to form pairwise stable networks within the symmetric connections model, whenever the expected payoff of adding or deleting a link is always higher than the expected payoff of deleting or adding a link, respectively. However, for certain parameter ranges, the action with the highest expected payoff will depend on the network structure. More information is need for agents using reinforcement learning to form these networks. My simulations yield evidence to suggest the network formation process will converge to either the empty network or the complete network depending on the parameters in the connections model. If I allow agents to observe the agent they are matched with and condition their actions based on this information, the network formation process may now converge to the pairwise stable networks ruled out in the first informational setting. This convergence is investigated through the use of simulations. The simulations suggest the network convergence to pairwise stable network occurs only when one action strictly dominates the other. A coordination problem arises when the star network is both pairwise stable and efficient.

The main assumption of this model is behavioral in nature. I assume a specific behavior for agents in order to allow for network formation with limited information. Testing this assumption in an economic laboratory is something that could be pursued in future research. Secondly, the assumption of symmetric costs and benefits in the connections model could be relaxed in future work.

CHAPTER IV

SUMMARY

In Chapter II, the role of individual incentives and competition in network formation was examined through a model in which two networks are competing for a completely divisible prize. Two well-known network allocation rules are examined: the Myerson value and the egalitarian rule. These allocation rules are both uniquely determined by a set of (different) axioms and offer different incentives. Several interesting results emerge when the winning probability for each network depends on the number of individuals which join each network. For certain values of the contest success function's parameter m, equal split divisions arise endogenously for the network which does not propose the equal split rule as a way to allocate the prize, rather the group is forced into an equal split by the presence of a competing network which divides the prize equally among its members. In other cases, the curvature of the contest success function creates an incentive for individuals to move to the larger group and the only equilibria involve all individuals joining the same group. For values of $m < 1$, certain graph partitions always form equilibria for the group using the Myerson value. Namely, subgraphs of a certain size with an equal number of connections, including cliques, and the densest subgraph with a certain upper bound. Finding these subgraphs are related to several famous problems in graph theory which are NP-Complete. Therefore, enumerating all equilibria for this model is intractable. Furthermore, whenever $m < 1$, the winning probability for the group using the Myerson value has an upper bound. The group using the egalitarian rule has no such upper bound. However, if the contest success function is changed to a function of the number of connections in each network, this is no longer the case. Equilibria in the first case may remain equilibria for this model with some additional conditions on the intra-group connections.

In Chapter III, three different models of network formation were examined. In each model, agents have limited information about the structure of the network and form links over time based on experience. The individual payoff functions follow the Connections Model [9] which includes two parameters: one associated with benefit of direct and indirect links and one which represents the cost of maintaing a direct link. Depending on these parameters, different network structures will be stable and efficient. The long run behavior of each model was examined by simulations and found to be different in certain parameter regions. If the cost of forming links with others is higher than the benefit, the empty network emerges after some time, regardless of the information the agents have access to. Also, when the cost of forming links is much lower than the benefit of linking with others, the complete network has a high probability of forming. However, this probability is different for each model. In both of these scenarios, one action is always better than the other. There are also values of the the parameters in which the long-run behavior of each model is different. For these cases, the best action depends on the network structure and having access to different information will affect how well agents are able to learn the "best" actions.

Together, these essays provide insight into how information, individual incentives, and competition can affect network formation. Individual incentives in the presence of competition can force individuals with a small number of social ties to join competing groups that do not place as much emphasis on social connectivity. Likewise, if individuals have many established social ties with a competing group and this group will reward these ties, they may leave a successful group. In this way, individual incentives and competition can affect the network structures of the groups or organizations involved in competition. Information also plays a critical role in network formation. If individuals lack information about the structure of network, it is hard for them to modify their social ties in the most beneficial way. However, in some cases individuals can quickly discern the best action no matter how other individuals are connected and information plays no role. This occurs when one action dominates the other for any network configuration. Whenever this is not the case, the amount and type of information available to individuals has a strong effect of the types of networks which emerge.

REFERENCES

[1] V. Bala, S. Goyal, A noncooperative model of network formation, Econometrica 68 (2000) 1181–1229.

[2] A. Beggs, On the convergence of reinforcement learning, J. Economic Theory 122 (2005) 1–36.

[3] T. Borgers, R. Sarin, Learning through reinforcement and replicator dynamics, J. Economic Theory 77 (1997) 1-14.

[4] L. Busoniu, R. Babuska, B. De Schutter, A comprehensive survey of multiagent reinforcement learning, IEEE Transactions on Systems, Man, and Cybernetics, Part C: Applications and Reviews 38 (2008) 156–172.

[5] S. Gensemer, L. Hong, J. Kelly, Migration disequilibrium and specific division rules, Soc. Choice Welfare 15 (1998) 201–209.

[6] S. Gensemer, L. Hong, J. Kelly, Division rules and migration equilibria, J. Economic Theory 69 (1996) 104–116.

[7] J.D. Gibbons, Nonparametric Methods for Quantitative Analysis, Holt, Rineheart, and Winston, New York, NY, 1976.

[8] K. Hwan-Baik, S. Lee, Collective rent seeking with endogenous group sizes, Europ. J. Polit. Economy 13 (1997) 121–130.

[9] M. Jackson, A. Wolinsky, A strategic model of social and economic networks, J. Economic Theory 71 (1996) 44–74.

[10] M. Jackson, Social and Economic Networks, Princeton University Press, Princeton, NJ, 2008.

[11] M. Jackson, A. Van Den Nouweland, Strongly stable networks, Games Econ. Behav. 51 (2005) 420–444.

[12] M. Jackson, Allocation rules for network games, Games Econ. Behav. 51 (2005) 128–154.

[13] M. Jackson, A. Watts, The evolution of social and economic networks, J. Economic Theory 106 (2002) 265–295.

[14] M. Jackson, A survey of models of network formation: Stability and efficiency, in: Group Formation in Economics: Networks, Clubs, and Coalitions, G. Demange, M. Wooders, (Eds.), Cambridge University Press, Cambridge, UK, 2005, pp. 11–57.

[15] C. Johnson, R. Gilles, Spatial social networks, Rev. Econ. Design 5 (2000) 273–299.

[16] A. Kirman, N. Vriend, Evolving market structure: An ACE model of price dispersion and loyalty J. Econ. Dynam. Control 25 (2001) 459-502.

[17] K. Konrad, Strategy and Dynamics in Contests, Oxford University Press, New York, NY, 2009.

[18] D. Leslie, E.J. Collins, Individual q-learning in normal form games, SIAM J. Control Optim. 44 (2005) 495–514.

[19] R. Myerson, Graphs and cooperation in games, Math. Oper. Res. 2 (1977) 225–229.

[20] G. Pickard, I. Rahwan, W. Pan, M. Cebrian, R. Crane, et al., Time critical social mobilization, Science 334 (2011) 509–512.

[21] T.W. Sandholm, R.H. Crites, Multiagent reinforcement learning in the iterated prisoner's dilemma, Biosystems 37 (1996) 147-166.

[22] R. Sarin, F. Vahid, Predicting how people play games: A simple dynamic model of choice, Games Econ. Behav. 34 (2001) 104–122.

[23] S. Skaperdas, Contest success functions, Econ. Theory 7 (1996) 283–290.

[24] B. Skyrms, R. Pemantle, A dynamic model of social network formation, Proc. Nat. Acad. Sciences 97 (2000) 9340-9346.

[25] M. Slikker, A. Van den Nouweland, Social and Economic Networks in Cooperative Game Theory, Kluwer Academic Publishers, Boston, MA, 2001.

[26] G. Tullock, Efficient rent seeking, in: Toward a Theory of the Rent-Seeking Society, J. M. Buchanan, R. D. Tollison, G. Tullock, (Eds.), Texas A&M University Press, College Station, TX, 1980, pp. 97-112.

[27] L. Waltman, U. Kaymak, Q-learning agents in a Cournot oligopoly model, J. Econ. Dynam. Control 32 (2008) 3275-3293.

[28] C. Watkins, P. Dayan, Q-learning, Machine Learning 8 (1992) 279–292.

[29] A. Watts, A dynamic model of network formation, Games Econ. Behav. 34 (2001) 331–341.